Equitable Partitioning Policies for Robotic Networks

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***Abstract*— The most widely applied resource allocation strategy is to balance, or equalize, the total workload assigned to each resource. In mobile multi-agent systems, this principle directly leads to equitable partitioning policies in which (i) the workspace is divided into subregions of equal measure, (ii) each agent is assigned to a unique subregion, and (iii) each agent is responsible for service requests originating within its own subregion. In this paper, we provide the first decentralized algorithm that provably allows m agents to converge to an equitable partition of the workspace, from any initial configuration, i.e. globally. Moreover, we extend our algorithms to take into account other relevant issues, like the desire to achieve an equitable partition in which the shapes of the subregions are as close as possible to regular polygons. Our approach is based on a modified gradient algorithm and provides novel insights into the properties of Power Diagrams. We discuss possible applications to dynamic vehicle routing and mobile sensor networks, and we provide extensive simulation results that show the effectivity of our algorithms.**

I. Introduction

The most widely applied resource allocation strategy is to balance, or equalize, the total workload assigned to each resource. While, in principle, several strategies are able to guarantee workload-balancing in multi-agent systems, *equitable partitioning policies* are predominant [1], [2], [3], [4]. A *partitioning policy* is an algorithm that, as a function of the number m of agents and, possibly, of their position and other information, partitions a bounded workspace A into subregions A_i, for $i \in \{1, \dots, m\}$. Then, each agent i is assigned to subregion A_i, and each service request in A_i receives service from the agent assigned to A_i. Accordingly, if we model the *workload* for subregion $S \subseteq A$ as $\lambda_S \doteq \int_S \lambda(x)\, dx$, where $\lambda(x)$ is a measure over A, then the workload for agent i is λ_{A_i}. Then, load balancing calls for equalizing the workload λ_{A_i} in the m subregions or, in equivalent words, requires to compute an *equitable partition* of the workspace A (i.e., a partition in subregions with the same measure).

Equitable partitioning policies are predominant for three main reasons: (i) efficiency, (ii) ease of design, (iii) ease of analysis. Consider, for instance, the well-known dynamic version of the classic Vehicle Routing Problem: the Dynamic Traveling Repairman Problem (DTRP) [1]. In the DTRP, m agents operating in a workspace A must service demands whose time of arrival, location and on-site service are stochastic; the obiective is to find a policy to service demands over an infinite horizon that minimizes the expected system time (wait plus service) of the demands. Equitable partitioning policies are, indeed, optimal for the DTRP (see [1], [5], [6]). As a second example, consider high-performance hybrid wireless networks: a hybrid network is formed by placing a sparse network of supernodes ("the agents") in an ad hoc network ("the customers"). Supernodes are more sophisticated and act as relays for normal nodes. As described in [3], an effective configuration for supernodes in hybrid networks is an equipartition configuration, where the subregions have the shape of regular hexagons. Equitable partitioning policies are, therefore, ubiquitous in multi-agent system applications.

Despite their relevance in robotic network applications, to the best of our knowledge, the only available *decentralized* equitable partitioning policy is the one proposed by the authors in [7]. However, the policy presented in [7] only guarantees *local* (i.e., for a subset of initial conditions) convergence to equitable partitions.

Building upon our previous work [7], in this paper we design distributed and adaptive policies that allow a team of agents to achieve *globally*, i.e. starting from any initial condition, a partition into subregions of equal measure.

The second contribution of this paper is to provide extensions of our algorithms to take into account *secondary* objectives, as for example, control on the shapes of the subregions. Our motivation, here, is that equitable partitions in which subregions are thin slices are, in most applications, useless: in the case of dynamic vehicle routing, for example, a thin slice partition would directly lead to an increase in fuel consumption.

We, finally, mention that our algorithms, although motivated in the context of multi-agent systems, are a novel contribution to the field of computational geometry; moreover, our results provide new insights in the geometry of Power Diagrams partition.

II. Background

In this section, we introduce some notation and briefly review some concepts from calculus and locational optimization, on which we will rely extensively later in the paper.

A. Notation

Let $\|\cdot\|$ denote the Euclidean norm. Let A be a compact, convex subset of $\mathbb{R}^d$. We denote the boundary of A as ∂A. The distance from a point x to a set M is defined as $\text{dist}(x, M) \doteq \inf_{p \in M} \|x - p\|$. We define $I_m \doteq \{1, 2, \cdots, m\}$. Let $G = (g_1, \cdots, g_m) \subset (A)^m$ denote the location of m points. A *partition* (or tessellation) of A is a collection of m closed subsets $\mathcal{A} = \{A_1, \cdots, A_m\}$ with

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disjoint interiors whose union is A. The partition of A is *convex*, if each subset A_i, $i \in I_m$, is convex. Finally, we define the saturation function $\text{sat}_{a,b}$, with $a < b$, as:

$$\text{sat}_{a,b} = \begin{cases} 1 & \text{if } x > b \\ (x-a)/(b-a) & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

B. Voronoi diagrams and Power Diagrams

We refer the reader to [8] and [9] for comprehensive treatments, respectively, of Voronoi diagrams and Power Diagrams. Assume that G is an ordered set of distinct points. The *Voronoi Diagram* $\mathcal{V}(G) = (V_1(G), \cdots, V_m(G))$ of A generated by points $G = (g_1, \cdots, g_m)$ is defined by

$$V_i(G) = \{x \in A |\, \|x - g_i\| \le \|x - g_j\|, \forall j \neq i,\ j \in I_m\}. \tag{2}$$

We refer to G as the set of *generators* of $\mathcal{V}(G)$, and to $V_i(G)$ as the Voronoi cell of the i-th generator. For $g_i, g_j \in G$, $i \neq j$, let $b(g_i, g_j) = \{x |\, \|x - g_i\| = \|x - g_j\|\}$ be the bisector of g_i and g_j; face $b(g_i, g_j)$ bisects the line segment joining g_i and g_j, and this line segment is orthogonal to the face (*Perpendicular Bisector Property*). It is easy to verify that each Voronoi cell is a *convex* set.

Assume, now, that each generator $g_i \in G$ has assigned an individual weight $w_i \in \mathbb{R}$, $i \in I_m$. We define $W = (w_1, \cdots, w_m)$. In some sense, w_i measures the capability of g_i to influence its neighborhood. This is expressed by the power distance $d_P(x, g_i; w_i) \doteq \|x - g_i\|^2 - w_i$.

We refer to the pair (g_i, w_i) as a *power point*. We define $G_W = \Big((g_1, w_1), \cdots, (g_m, w_m)\Big) \in (A \times \mathbb{R})^m$. Two power points (g_i, w_i) and (g_j, w_j) are *coincident* if $g_i = g_j$ *and* $w_i = w_j$. Assume that G_W is an ordered set of *distinct* power points. Similarly as before, the *Power Diagram* $\mathcal{V}(G_W) = (V_1(G_W), \cdots, V_m(G_W))$ generated by power points $G_W = \Big((g_1, w_1), \cdots, (g_m, w_m)\Big)$ is defined by

$$V_i(G_W) = \{x \in A |\, \|x - g_i\|^2 - w_i \le \|x - g_j\|^2 - w_j, \\ \forall j \neq i,\ j \in I_m\}. \tag{3}$$

We refer to G_W as the set of *power generators* of $\mathcal{V}(G_W)$, and to $V_i(G_W)$ as the power cell of the i-th power generator; moreover we call g_i and w_i, respectively, the position and the weight of power generator (g_i, w_i). Notice that, when all weights are the same, the Power Diagram coincides with the Voronoi Diagram. Each power cell is, as well, a convex set (as it can be easily verified). Notice that (i) a power cell might be empty, and (ii) g_i might not be in its power cell. Finally, the bisector of (g_i, w_i) and (g_j, w_j), $i \neq j$, is defined as

$$b\Big((g_i, w_i), (g_j, w_j)\Big) = \{x \in A |\, (g_j - g_i)^{\mathrm{T}} x = \\ \frac{1}{2}(\|g_j\|^2 - \|g_i\|^2 + w_i - w_j)\}. \tag{4}$$

Hence, $b\Big((g_i, w_i), (g_j, w_j)\Big)$ is a face orthogonal to the line segment $\overline{g_i\, g_j}$. Notice that the Power diagram of an ordered set of possibly *coincident* power points is not well-defined. We define

$$\Gamma_{\text{coinc}} = \Big\{G_W |\, g_i = g_j \text{ and } w_i = w_j \text{ for some } i \neq j \in I_m\Big\}. \tag{5}$$

For simplicity, we will refer to $V_i(G)$ ($V_i(G_W)$) as V_i. When the two Voronoi (power) cells V_i and V_j are adjacent (i.e., they share a face), g_i ((g_i, w_i)) is called a *Voronoi* (*power*) *neighbor* of g_j ((g_j, w_j)), and vice-versa. The set of indices of the Voronoi (power) neighbors of g_i ((g_i, w_i)) is denoted by N_i. We also define the (i, j)-face as $\Delta_{ij} \doteq V_i \cap V_j$.

C. A Basic Result in Homotopy Theory

Given two topological manifolds, X, Y and given two continuous maps $f, g : X \to Y$, we say that f is homotopic to g if there exists a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. One can prove the following.

Theorem 2.1: Let $f : C \to B$ a continuous map from a closed cube $C \subset \mathbb{R}^p$ to a closed ball $B \subset \mathbb{R}^q$. Assume that the map is surjective on the boundary of B, ∂B, meaning that $\partial B \subset f(C)$. Then f is surjective.

III. Problem Formulation

A total of m identical mobile agents provide service in a compact, convex service region $A \subseteq \mathbb{R}^d$. Let λ be a measure whose bounded support is A (in equivalent words, λ is not zero only on A); for any set S, we define the *workload* for region S as $\lambda_S \doteq \int_S \lambda(x)\, dx$. The measure λ models service requests, and can represent, for example, the density of customers over A, or, in a stochastic setting, their arrival rate. Given the measure λ, a partition $\{A_i\}_i$ of the workspace A is *equitable* if $\lambda_{A_i} = \lambda_{A_j}$ for all $i, j \in I_m$.

A *partitioning policy* is an algorithm that, as a function of the number m of agents and, possibly, of their position and other information, partitions a bounded workspace A into subregions A_i, $i \in I_m$. Then, each agent i is assigned to subregion A_i, and each service request in A_i receives service from the agent assigned to A_i. We refer to subregion A_i as the *region of dominance* of agent i. Given a measure λ and a partitioning policy, m agents are in a *convex equipartition configuration* with respect to λ if the associated partition is equitable and convex.

In this paper we are interested in the following problem: find distributed equitable partitioning policies that allow m mobile agents to *globally* (i.e., from any initial condition) reach a convex equipartition configuration (with respect to λ). Moreover, we consider the issue of convergence to equitable partitions with some special properties, e.g., where subregions have shapes similar to regular polygons.

IV. On the existence of equitable Power Diagrams

The key advantage of Power Diagrams is that an equitable Power Diagram always exists for any λ. Indeed, as shown in the next theorem, an equitable Power Diagram (with respect to any λ) exists for *any* vector of *distinct* points $G = (g_1, \dots, g_m)$ in A.

Theorem 4.1: Let A be a bounded, connected domain in $\mathbb{R}^2$, and λ be a measure on A. Let $G = (g_1, \ldots, g_m)$ be the positions of $m \geq 1$ distinct points in A. Then, there exist weights w_i, $i \in I_m$, such that the power points $\Big((g_1, w_1), \ldots, (g_m, w_m)\Big)$ generate a Power Diagram that is equitable with respect to λ. Moreover, given a vector of weights W^* that yields an equitable partition, the set of all vectors of weights yielding an equitable partition is $\mathcal{W} \doteq \{W^* + t[1, \ldots, 1]\}$, with $t \in \mathbb{R}$.

Proof: It is not restrictive to assume that $\lambda_A = 1$ (i.e. we normalize the measure of A). First, we construct a weight space. Let $D = \mathrm{diameter}(A)$, and consider the cube $\mathcal{C} := [-D, D]^m$. This is the weight space and we consider weight vectors W taking value in $\mathcal{C}$. Second, consider the standard *m-simplex* of measures $\lambda_{A_1}, \ldots, \lambda_{A_m}$ (where $A_1, \ldots, A_m$ are, as usual, the regions of dominance). This can be realized in $\mathbb{R}^m$ as the subset of defined by $\sum_{i=1}^m \lambda_{A_i} = 1$ with the condition $\lambda_{A_i} \geq 0$. Let's call this set "the measure simplex $\mathcal{A}$" (notice that it is $(m-1)$-dimensional).

There is a map $f : \mathcal{C} \to \mathcal{A}$ associating, according to the power distance, a weight vector W with the corresponding vector of measures $(\lambda_{A_1}, \ldots, \lambda_{A_m})$. Since the points in G are assumed to be distinct, this map is continuous. We will now use induction on m, starting with the base case $m = 3$ (the statement for $m = 1$ and $m = 2$ is trivial). When $m = 3$, the weight space $\mathcal{C}$ is a three dimensional cube with vertices $v_0 = [-D, -D, -D]$, $v_1 = [D, -D, -D]$, $v_2 = [-D, D, -D]$, $v_3 = [-D, -D, D]$, $v_4 = [D, -D, D]$, $v_5 = [-D, D, D]$, $v_6 = [D, D, -D]$ and $v_7 = [D, D, D]$. The measure simplex $\mathcal{A}$ is, instead, a triangle with vertices u_1, u_2, u_3 that correspond to the cases $\lambda_{A_1} = 1, \lambda_{A_2} = 0, \lambda_{A_3} = 0$, $\lambda_{A_1} = 0, \lambda_{A_2} = 1, \lambda_{A_3} = 0$, and $\lambda_{A_1} = 0, \lambda_{A_2} = 0, \lambda_{A_3} = 1$, respectively. Moreover, call e_1, e_2 and e_3 the edges opposite the vertices u_1, u_2, u_3 respectively. The edges e_i are, therefore, given by the condition $\lambda_{A_i} \in e_i \Leftrightarrow \lambda_{A_i} = 0$.

Let's return to the map $f : \mathcal{C} \to \mathcal{A}$ (now in the case of three generators). Observe that the map f sends v_0 the the unique point p_0 of $\mathcal{A}$ corresponding to the measures of usual Voronoi cells (since the weights are all equal). Call l_1 the edge $v_0 v_1$; then, it is immediate to see that image of l_1 through f is a path γ_1 in $\mathcal{A}$ joining p_0 to u_1.

Analogously, the image of $l_2 = v_0 v_2$ through f is a path γ_2 in $\mathcal{A}$ joining p_0 to u_2 and, finally, the image of $l_3 = v_0 v_3$ through f is a path γ_3 connecting p_0 to u_3. It is easy to see that paths $\{\gamma_i\}_{\{i=1,2,3\}}$ do not intersect except in p_0. Since the measures of the regions of dominance do not change if the differences among the weights are kept constant, then it is easy to see that the fibers of f in the weight space $\mathcal{C}$ are exactly straight lines *parallel* to the main diagonal $v_0 v_7$. On the weight space $\mathcal{C}$ let us define the following equivalence relation: $w \equiv w'$ if and only if they are on a line parallel to the main diagonal $v_0 v_7$. Map $f : \mathcal{C} \to \mathcal{A}$ induces a continuous map (still called f by abuse of notation) from $\mathcal{C}/\equiv$ to $\mathcal{A}$ having the same image. Let us identify $\mathcal{C}/\equiv$. It is easy to see that any line in the cube parallel to the main diagonal $v_0 v_7$ is entirely determined by its intersection with the three faces $F_3 = \{w_3 = -D\} \cap \mathcal{C}$, $F_2 = \{w_2 = -D\} \cap \mathcal{C}$ and $F_1 = \{w_1 = -D\} \cap \mathcal{C}$. Call $\mathcal{F}$ the union of these faces. We therefore have a continuous map $f : \mathcal{F} \to \mathcal{A}$ that has the same image of our original f. Now $f : \mathcal{F} \to \mathcal{A}$ is *injective* by construction and has the same image of $f : \mathcal{C} \to \mathcal{A}$. Therefore, since $\gamma_i = f(l_i)$, and edges l_i have only one common point in $\mathcal{F}$, namely vertex v_0, their images γ_i can only have p_0 as a common point in $\mathcal{A}$, since by construction $f : \mathcal{F} \to \mathcal{A}$ is injective. This proves that paths γ_i do not intersect except in p_0.

Therefore, the median point of $\mathcal{A}$, call it p^* must lie in one of the regions in which the triangle $\mathcal{A}$ is divided by the paths γ_i. If it falls on one of the paths there is nothing to prove. It is not restrictive to assume that p^* lies in the interior of the region whose boundary is described by γ_3, γ_2 and the edge e_1. Consider the path Γ in $\mathcal{A}$ beginning and ending at p_0 given by γ_2, then the edge e_1 and then the path γ_3. It encloses p^* by assumption.

We already know that γ_2 and γ_3 are the images through f of the edges l_2 and l_3 respectively. Consider now the closed loop α on the boundary of $\mathcal{C}$ starting at v_0 and going thorough the edges $l_2 = v_0 v_2$, $v_2 v_5$, $v_5 v_3$, $v_3 v_0 = l_3$. It is easy to see that f maps this closed path to the path Γ. Parametrize the path α using any continuos map $\alpha : S^1 \to \mathcal{C}$, so that the image of this map is the path α itself. (Here S^1 is the unit circle.) We, then, have a map $\Gamma := f \circ \alpha : S^1 \to \mathcal{A}$, whose image is the path Γ itself. Since $\mathcal{C}$ is contractible there exists a continuous homotopy $H : S^1 \times [0, 1] \to \mathcal{C}$, such that $H(t, 0) = \alpha(t)$ and $H(t, 1) = V_0$. Now, composing H with f, we get a homotopy $K := f \circ H : S^1 \times [0, 1] \to \mathcal{A}$ such that $K(t, 0) = \Gamma(t)$ and $K(t, 1) = P_0$. This implies that the path Γ can be continuously shrank to the point p_0, so f must be surjective and p^* belongs to its image. This proves that map $f : \mathcal{C} \to \mathcal{A}$ is surjective for $m = 3$.

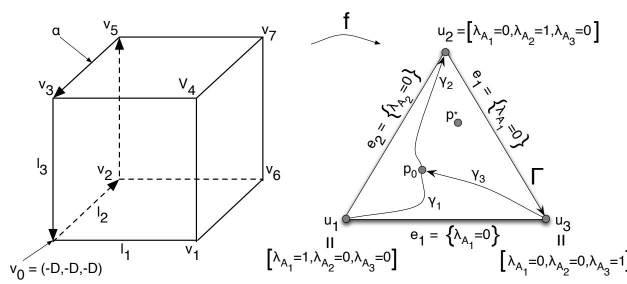


Fig. 1. Construction used for the proof of existence of equitable Power Diagrams.

The case with m points is based on an inductive argument that will not be developed for the sake of space. The basic idea is again to use topological forcing to prove existence. The details and second part of the Theorem are left to the reader. ■

Remark 4.2: Since all vectors of weights in $\mathcal{W}$ yield exactly the *same* Power Diagram, we conclude that the positions of the generators *uniquely* induces an equitable Power Diagram.

V. Gradient Descent Law for Equitable Partitioning

In this section, we design distributed policies that allow a team of agents to achieve a convex equipartition configuration.

A. Virtual Generators

The first step is to associate to each agent i a *virtual power generator* (virtual generator for short) (g_i, w_i). We define the region of dominance for agent i as the Power cell $V_i = V_i(G_W)$, where $G_W = \Big((g_1, w_1), \cdots, (g_m, w_m)\Big)$ (see Fig. 2). We refer to the partition into regions of dominance induced by the set of virtual generators G_W as $\mathcal{V}(G_W)$. A virtual generator (g_i, w_i) is simply an artificial variable locally controlled by the i-th agent; in particular, g_i is a virtual point and w_i is its weight.

Virtual generators allow us to decouple the problem of achieving an equitable partition into regions of dominance from that of positioning an agent inside its own region of dominance. We shall assume that each vehicle has sufficient information available to determine: (1) its Power cell, and (2) the locations of all outstanding events in its Power cell. A control policy that relies on information (1) and (2), is *distributed* in the sense that the behavior of each vehicle depends only on the location of the other agents with contiguous Power cells. A spatially distributed algorithm for the local computation and maintenance of Power cells can be designed following the ideas in [10].

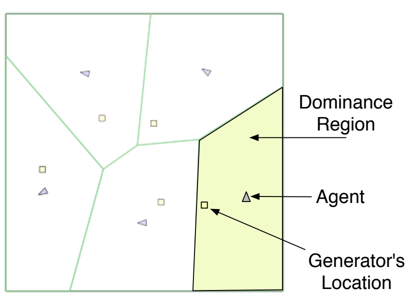


Fig. 2. Agents, virtual generators and regions of dominance.

B. Locational Optimization

In light of Theorem 4.1, the key idea is to enable the weights of the virtual generators to follow a (distributed) gradient descent law (while maintaining the positions of the generators *fixed*) such that an equitable partition is reached.

Assume, henceforth, that the positions of the virtual generators are *distinct*, i.e. $g_i \neq g_j$ for $i \neq j$. Define the set

$$S \doteq \Big\{(w_1, \dots, w_m) \in (\mathbb{R})^m \mid \lambda_{V_i} > 0 \;\; \forall i \in I_m\Big\}. \tag{6}$$

Set S contains all possible vectors of weights such that no region of dominance has measure *zero*.

We introduce the locational optimization function $H_V : S \mapsto \mathbb{R}_{>0}$:

$$H_V(W) \doteq \sum_{i=1}^{m}\Big(\int_{V_i}\lambda(x)dx\Big)^{-1} = \sum_{i=1}^{m}\lambda_{V_i}^{-1}. \tag{7}$$

C. Smoothness and Gradient of H_V

We now analyze the smoothness properties of the locational optimization function H_V. In the following, let δ_{ij} be the length of the border Δ_{ij}, and let $\gamma_{ij} \doteq \|g_j - g_i\|$.

Theorem 5.1: Assume that the positions of the virtual generators are *distinct*, i.e. $g_i \neq g_j$ for $i \neq j$. Given a measure λ, the locational optimization function H_V is continuously differentiable on S, where for each $i \in \{1, \dots, m\}$

$$\frac{\partial H_V}{\partial\, w_i} = \sum_{j\in N_i}\frac{1}{2\gamma_{ij}}\Big(\frac{1}{\lambda_{V_j}^2} - \frac{1}{\lambda_{V_i}^2}\Big)\int_{\Delta_{ij}}\lambda(x)\,dx. \tag{8}$$

Furthermore, the critical configurations of H_V are generators' weights with the property that all power cells have measure equal to λ_A/m.

Proof: The proof is almost identical to that of Theorem 5.1 in [7] and, thus, it is omitted. ■

Remark 5.2: The gradient in Theorem 5.1 can be computed in a distributed way, since it depends only on the location of the other agents with contiguous Power cells.

Example 5.3 (Gradient of H_V for uniform measure): The gradient of H_V simplifies considerably when λ is constant. In such case it is straightforward to verify that (assuming that λ is normalized)

$$\frac{\partial H_V}{\partial\, w_i} = \frac{1}{2|A|}\sum_{j\in N_i}\frac{\delta_{ij}}{\gamma_{ij}}\Big(\frac{1}{|V_j|^2} - \frac{1}{|V_i^2|}\Big). \tag{9}$$

D. Spatially-Distributed Algorithm for Equitable Partitioning

Consider the set $U \doteq \Big\{(w_1, \dots, w_m) \in (\mathbb{R})^m \mid \sum_{i=1}^m w_i = 0\Big\}$. Indeed, since adding an identical value to every weight leaves all power cells unchanged, there is *no loss of generality* in restricting the weights to U; let $\Omega \doteq S \cap U$. Assume the generators' weight obey a first order dynamical behavior described by $\dot{w}_i = u_i$. Consider H_V an aggregate objective function to be minimized and impose that the weight w_i follows the gradient descent given by (8). In more precise terms, we set up the following control law defined over the set Ω

$$u_i = -\frac{\partial H_V}{\partial w_i}(W), \tag{10}$$

where we assume that the Power Diagram $\mathcal{V}(W) = \{V_1, \dots, V_m\}$ is continuously updated. One can prove the following result.

Theorem 5.4: Assume that the positions of the virtual generators are *distinct*, i.e. $g_i \neq g_j$ for $i \neq j$. Consider the gradient vector field on Ω defined by equation (10). Then generators' weights starting at $t = 0$ at $W(0) \in \Omega$, and evolving under (10) remain in Ω and converge asymptotically to a critical point of the aggregate objective function H_V, i.e. to a vector of weights that yields an equitable Power diagram.

Proof: We first prove that generators' weights evolving under (10) remain in Ω and converge asymptotically to the *set* of critical points of the aggregate objective function H_V. By assumption, $g_i \neq g_j$ for $i \neq j$, thus the Power diagram is well defined. First, we prove that set Ω is positively invariant with respect to (10). Recall that $\Omega = S \cap U$. Noticing that control law (10) is a gradient descent law, we have

$$\lambda_{V_i}^{-1} \leq H_V(W(t)) \leq H_V(W(0)), \quad i \in I_m,\ t \geq 0.$$

Since the measures of the power cells depend continuously on the weights, we conclude that the measures of all power cells will be bounded away from zero; thus, the weights will belong to S for all $t \geq 0$, i.e. $W(t) \in S\ \forall t \geq 0$. Moreover, the sum of the weights is invariant under control law (10). Indeed,

$$\frac{\partial \sum_{i=1}^m w_i}{\partial t} = -\sum_{i=1}^m \frac{\partial H_V}{\partial w_i} = -\sum_{i=1}^m \sum_{j\in N_i} \frac{1}{2\gamma_{ij}}\Big(\frac{1}{\lambda_{V_j}^2} - \frac{1}{\lambda_{V_i}^2}\Big) \int_{\Delta_{ij}} \lambda(x)\, dx = 0,$$

since $\gamma_{ij} = \gamma_{ji}$, $\Delta_{ij} = \Delta_{ji}$, and $j \in N_i \Leftrightarrow i \in N_j$. Thus, we have $W(t) \in U\ \forall t \geq 0$. Since $W(t) \in S\ \forall t \geq 0$ and $W(t) \in U\ \forall t \geq 0$, we conclude that $W(t) \in S \cap U = \Omega$, $\forall t \geq 0$, i.e. set Ω is positively invariant.

Second, $H_V : \Omega \mapsto \mathbb{R}_{\geq 0}$ is clearly non-increasing along system trajectories, i.e. $\dot{H}_V(W) \leq 0$ in Ω.

Third, all trajectories with initial conditions in Ω are bounded. Indeed, we have already shown that $\sum_{i=1}^m w_i = 0$ along system trajectories. This implies that weights remain within a bounded set: If, by contradiction, a weight could become arbitrarily large, another weight would become arbitrarily small (since the sum of weights is constant), and the measure of at least one power cell would vanish, which contradicts the fact that S is positively invariant.

Finally, by Theorem 5.1, H_V is continuously differentiable in Ω. Hence, by invoking the LaSalle invariance principle, under the descent flow (10), weights will converge asymptotically to the *set* of critical points of H_V, that is not empty by Theorem 4.1.

Indeed, by Theorem 4.1, we know that all vectors of weights yielding an equitable partition differ by a common translation. Thus, the largest invariant set of H_V is Ω contains only one point. This implies that $\lim_{t\to\infty} W(t)$ exists and it is equal to a vector of weights that yields an equitable Power diagram.

■

Some remarks are in order.

Remark 5.5: Since, by Theorem 5.1, all critical configurations of H_V are generators' weights with the property that all power cells have measure equal to λ_A/m, *convergence to an equitable partition is global* with respect to Ω. Indeed, there is a very natural choice for the initial values of the virtual power generators. Assuming that at $t = 0$ agents are in A and in distinct positions, each agent initializes the position of its virtual generator to its current position, and initializes its weight to zero. Then, the initial partition is a Voronoi tessellation; since λ is positive on A, each initial cell has nonzero measure, and therefore $W(0) \in \Omega$ (the sum of the initial weights is clearly zero).

Remark 5.6: As discussed before, by adopting the algorithm in [10], each agent can compute its Power cell in a distributed way. Moreover, the partial derivative of H_V with respect to the i-th weight only depends on the weights of virtual generators with neighboring Power cells. Therefore, the gradient descent law (10) is indeed a distributed control law. We mention that, in a Power Diagram, each generator has an average number of neighbors less then six [11]; therefore, the computation of gradient (10) is scalable with the number of agents.

■

VI. Distributed Algorithms for Equitable Partitions with Special Properties

The previous gradient descent law, although effective in providing a convex equitable partition, can yield long and "skinny" subregions. In this section we provide algorithms to obtain convex equitable partitions while optimizing a secondary objective function. The key idea is that, to obtain an equitable partition, changing the weights, while maintaining the generators *fixed*, is sufficient. Thus, we can use the degrees of freedom given by the locations of the generators to optimize secondary cost functionals.

Specifically, define the set

$$\tilde{S} \doteq \Big\{\Big((g_1, w_1), \ldots, (g_m, w_m)\Big) \in (A \times \mathbb{R})^m \mid g_i \neq g_j \text{ for all } i \neq j, \text{ and } \lambda_{V_i} > 0\ \forall i \in I_m\Big\}. \tag{11}$$

We now assume that both generators' weights and positions obey a first order dynamical behavior described by $\dot{w}_i = u_i^w$ and $\dot{g}_i = u_i^g$. The *primary* objective is to achieve a convex equitable partition and is captured, similarly as before, by the cost functional $\tilde{H}_V : \tilde{S} \mapsto \mathbb{R}_{>0}$: $\tilde{H}_V(G_W) \doteq \sum_{i=1}^m \lambda_{V_i}^{-1}$.

We have the following

Theorem 6.1: Given a measure λ, the primary objective function $\tilde{H}_V$ is continuously differentiable on $\tilde{S}$, where for each $i \in \{1, \ldots, m\}$

$$\begin{aligned}\frac{\partial \tilde{H}_V}{\partial g_i} &= \sum_{j\in N_i} \Big(\frac{1}{\lambda_{V_j}^2} - \frac{1}{\lambda_{V_i}^2}\Big) \int_{\Delta_{ij}} \frac{(x - g_i)}{\gamma_{ij}} \lambda(x)\, dx, \\ \frac{\partial \tilde{H}_V}{\partial w_i} &= \sum_{j\in N_i} \Big(\frac{1}{\lambda_{V_j}^2} - \frac{1}{\lambda_{V_i}^2}\Big) \int_{\Delta_{ij}} \frac{1}{2\gamma_{ij}} \lambda(x)\, dx.\end{aligned} \tag{12}$$

Furthermore, the critical configurations of $\tilde{H}_V$ are generators' locations and weights with the property that all power cells have measure equal to λ_A/m.

Proof: The proof of this Theorem is very similar to the proof of Theorem 5.1; we omit it in the interest of brevity.

■

The gradient in Theorem 6.1 can be computed in a distributed way. For short, we define the vectors $v_{\pm\partial\tilde{H}_i} \doteq \pm\frac{\partial \tilde{H}_V}{\partial g_i}$. Three possible secondary objectives are discussed in the remainder of this section.

A. Optimizing the centroidal defect

Define the *mass* and the *centroid* of the Power cell V_i, $i \in I_m$, as $M_{V_i} = \int_{V_i} \lambda(x)\, dx$, and $C_{V_i} = \frac{1}{M_{V_i}} \int_{V_i} x\lambda(x)\, dx$. In this section we are interested in the situation where $g_i = C_{V_i}$, for all $i \in I_m$. We call such a Power Diagram a *centroidal Power Diagram*. The main motivation to study centroidal

Power Diagram is that, as it will be extensively discussed in Sec. VI-C, their cells, under certain conditions, are *close* in shape to regular hexagons.

A natural way to try to obtain a centroidal Power Diagram (or at least a *good* approximation of it) is to let the positions of the generators move toward the centroids of the corresponding regions of dominance, when this motion does not increase the disagreement between the measures of the cells (i.e. it does not make the time derivative of $\tilde{H}_V$ positive). First we introduce a $\mathcal{C}^\infty$ saturation function as follows:

$$F(x) \doteq \begin{cases} 0 & \text{for } x \leq 0 \\ \exp(-\frac{1}{x^2}) & \text{for } x > 0 \,. \end{cases} \tag{13}$$

Define the vector $v_{C,g_i} \doteq C_{V_i} - g_i$. Then, we set up the following control law defined over the set $\tilde{S}$, where we assume that the partition $\mathcal{V}(G_W) = \{V_1, \ldots, V_m\}$ is continuously updated,

$$\begin{aligned} \dot{w}_i &= -\frac{\partial \tilde{H}_V}{\partial w_i}, \\ \dot{g}_i &= v_{C,g_i} F(v_{C,g_i} \cdot v_{-\partial \tilde{H}_i}) \frac{2}{\pi} \arctan \left[\frac{\|v_{-\partial \tilde{H}_i}\|^2}{\alpha} \right] \end{aligned} \tag{14}$$

In other words, g_i moves toward the centroid of its cell if and only if this motion is compatible with the minimization of $\tilde{H}_V$. The term $F(v_{C,g_i} \cdot v_{-\partial \tilde{H}_i}) \arctan \|v_{-\partial \tilde{H}_i}\|^2/\alpha$ is needed to make the right hand side of (14) C^1 and compatible with the minimization of $\tilde{H}_V$. To prove that the vector field is C^1 it is simply sufficient to observe that it is the composition and product of C^1 functions. Furthermore, the compatibility condition of the flow (14) with the minimization of $\tilde{H}_V$ stems from the fact the $\dot{g}_i = 0$ as long as $v_{C,g_i} \cdot v_{-\partial \tilde{H}_i} \leq 0$, due to the presence of F. Notice that the right hand side of (14) can be computed in a distributed way.

As in many algorithms that involve the update of generators of Voronoi diagrams, it is possible that under control law (14) there exists a time t^* and $i, j \in I_m$ such that $g_i(t^*) = g_j(t^*)$. As noticed above, when two power generators coincide, either the Power diagram is not defined (when $w_i(t^*) = w_j(t^*)$), or there is an empty cell ($w_i(t^*) \neq w_j(t^*)$), and there is not obvious way to specify the behavior of the control laws for these singularity points. Thus, to make the set $\tilde{S}$ positively invariant, we have to modify slightly the update equation for the positions of the virtual generators. The idea is to stop two generators when they are *close* and on a *collision course*.

Define, for $\Delta \in \mathbb{R}_{>0}$, the set

$$M_i(G, \Delta) \doteq \{g_j \in G \,|\, \|g_i - g_j\| \leq \Delta, \, g_j \neq g_i\}.$$

In other words, M_i is the set of generators within an (Euclidean) distance Δ from g_i. For $\delta \in \mathbb{R}_{>0}$, $\delta < \Delta$, define the gain function $\psi(\rho, \vartheta) : [0, \Delta] \times [0, 2\pi] \mapsto \mathbb{R}_{\geq 0}$:

$$\begin{cases} \frac{\rho-\delta}{\Delta-\delta} & \text{if } \delta < \rho \leq \Delta \quad \wedge\, 0 \leq \vartheta < \pi, \\ \frac{\rho-\delta}{\Delta-\delta}(1 + \sin\vartheta) - \sin\vartheta & \text{if } \delta < \rho \leq \Delta \quad \wedge\, \pi \leq \vartheta \leq 2\pi, \\ 0 & \text{if } \rho \leq \delta \quad \wedge\, 0 \leq \vartheta < \pi, \\ -\frac{\rho}{\delta}\sin\vartheta & \text{if } \rho \leq \delta \quad \wedge\, \pi \leq \vartheta \leq 2\pi, \end{cases} \tag{15}$$

where $\wedge$ represents the logical "and". It is easy to see that $\psi(\cdot)$ is a continuous function on $[0, \Delta] \times [0, 2\pi]$ and it is globally Lipschitz there. Function $\psi(\cdot)$ has the following motivation. Let ρ be equal to $\|g_j - g_i\|$, and ϑ be the angle between vectors v_{C,g_i} and $(g_j - g_i)$. If $\rho \leq \delta$ and $0 \leq \vartheta < \pi$, then g_i is *close* to g_j and it is on a *collision course*, thus we set the gain to zero. Similar considerations hold for the other three cases; for example, if $\rho \leq \delta$ and $\pi < \vartheta < 2\pi$, the generators are *close*, but not on a collision course, thus the gain is positive.

Thus, we modify control law (14) as follows:

$$\begin{aligned} \dot{w}_i &= -\frac{\partial \tilde{H}_V}{\partial w_i} \doteq u_i^{\text{cent,w}}, \\ \dot{g}_i &= v_{C,g_i} F(v_{C,g_i} \cdot v_{-\partial \tilde{H}_i}) \frac{2}{\pi} \arctan \left[\frac{\|v_{-\partial \tilde{H}_i}\|^2}{\alpha} \right] \cdot \\ &\qquad \prod_{g_j \in M_i(G,\Delta)} \psi\Big(\|g_j - g_i\| . \measuredangle(v_{C,g_i}, g_j - g_i)\Big) \doteq u_i^{\text{cent,g}} \end{aligned} \tag{16}$$

where, $\measuredangle(v_{C,g_i}, g_j - g_i)$ denotes the angle between vectors v_{C,g_i} and $(g_j - g_i)$; if $M_i(G, \Delta)$ is the empty set, then we have an empty product, whose numerical value is 1. Notice that the right hand side of (16) is Lipschitz continuous, since it is a product of C^1 functions and Lipschitz continuous functions and it can be still computed in a distributed way (in fact, it only depends on generators that are neighbors in the Power diagram, and are within a distance Δ). One can prove the following result.

Theorem 6.2: Consider the vector field on $\tilde{S}$ defined by equation (16). Then generators' positions and weights starting at $t = 0$ at $G_W(0) \in \tilde{S}$, and evolving under (16) remain in $\tilde{S}$ and converge asymptotically to the set of critical points of the primary objective function $\tilde{H}_V$ (i.e. to the set of vectors of generators' positions and weights that yield an equitable Power diagram).

Proof: The proof is similar to that of Theorem 5.4 and we omit it in the interest of brevity. ■

B. Optimizing the Voronoi defect

In other applications it could be preferable to have a partition as *close* as possible to a Voronoi diagram. This issue is of particular interest for the setting with non-uniform density, when an equitable Voronoi diagram could fail to exist [7]. The objective of minimizing the Voronoi defect of a Power diagram can be translated in the minimization of the functional $K : \mathbb{R}^m \mapsto \mathbb{R}_{\geq 0}$: $K(W) \doteq \sum_{i=1}^m w_i^2$; when $w_i = 0$ for all $i \in I_m$, we have $K = 0$ and the corresponding Power diagram coincides with a Voronoi diagram. To include the minimization of the secondary objective K, it is natural to consider the following update law for the weights: $\dot{w}_i = -\frac{\partial \tilde{H}_V}{\partial w_i} - \frac{\partial K}{\partial w_i}$. However, $\tilde{H}_V$ is no longer a valid Lyapunov function for such system. The idea, then, is to let the positions of the generators move so that $\frac{\partial \tilde{H}_V}{\partial g_i} \dot{g}_i - \frac{\partial \tilde{H}_V}{\partial w_i} \frac{\partial K}{\partial w_i} = 0$. In other words, the dynamics of the generators is used to compensate the effect of the term $-w_i$ (present in the weights' dynamics) on the time derivative of

$\tilde{H}_V$. Thus, we set up the following control law, with ε_1, ε_2 and ε_3 positive *small* constants,

$$\begin{aligned}\dot{w}_i &= -\frac{\partial \tilde{H}_V}{\partial w_i} - w_i \mathrm{sat}_{\varepsilon_1,\varepsilon_2}\Big(\|v_{\partial \tilde{H}_i}\|\Big)\, \mathrm{sat}_{0,\varepsilon_3}\Big(\mathrm{dist}(g_i, \partial V_i)\Big),\\ \dot{g}_i &= w_i \frac{\partial \tilde{H}_V}{\partial w_i}\frac{v_{\partial \tilde{H}_i}}{\|\ v_{\partial \tilde{H}_i}\|^2}\, \mathrm{sat}_{\varepsilon_1,\varepsilon_2}\Big(\|v_{\partial \tilde{H}_i}\|\Big)\, \mathrm{sat}_{0,\varepsilon_3}\Big(\mathrm{dist}(g_i, \partial V_i)\Big)\end{aligned} \tag{17}$$

the gain $\mathrm{sat}_{\varepsilon_1,\varepsilon_2}\Big(\|v_{\partial H_i}\|\Big)$ is needed to make the right hand side of (17) Lipschitz continuous, while the gain $\mathrm{sat}_{0,\varepsilon_3}\Big(\mathrm{dist}(g_i, \partial V_i)\Big)$ avoids generators leaving the workspace. Notice that the right hand side of (17) can be computed in a distributed way. As before, it is possible that under control law (17) there exists a time t^* and $i, j \in I_m$ such that $g_i(t^*) = g_j(t^*)$ and $w_i(t^*) = w_j(t^*)$. Thus, similarly as before, we modify the update equations (17) as follows (where $v_{g_j,g_i} \doteq g_j - g_i$):

$$\begin{aligned}\dot{w}_i &= -\frac{\partial \tilde{H}_V}{\partial w_i} - w_i \mathrm{sat}_{\varepsilon_1,\varepsilon_2}\Big(\|v_{\partial \tilde{H}_i}\|\Big)\, \mathrm{sat}_{0,\varepsilon_3}\Big(\mathrm{dist}(g_i, \partial V_i)\Big)\cdot\\ &\prod_{g_j\in M_i(G,\Delta)} \psi\Big(\|v_{g_j,g_i}\|, \measuredangle(w_i \frac{\partial \tilde{H}_V}{\partial w_i} v_{\partial \tilde{H}_i}, v_{g_j,g_i})\Big) \doteq u_i^{\text{vor,w}},\\ \dot{g}_i &= w_i \frac{\partial \tilde{H}_V}{\partial w_i}\frac{v_{\partial \tilde{H}_i}}{\|\ v_{\partial \tilde{H}_i}\|^2} \mathrm{sat}_{\varepsilon_1,\varepsilon_2}\Big(\|v_{\partial \tilde{H}_i}\|\Big)\, \mathrm{sat}_{0,\varepsilon_3}\Big(\mathrm{dist}(g_i, \partial V_i)\Big)\cdot\\ &\prod_{g_j\in M_i(G,\Delta)} \psi\Big(\|v_{g_j,g_i}\|, \measuredangle(w_i \frac{\partial \tilde{H}_V}{\partial w_i} v_{\partial \tilde{H}_i}, v_{g_j,g_i})\Big) \doteq u_i^{\text{vor,g}}\end{aligned} \tag{18}$$

One can prove the following result.

Theorem 6.3: Consider the gradient vector field on $\tilde{S}$ defined by equation (18). Then generators' positions and weights starting at $t = 0$ at $G_W(0) \in \tilde{S}$, and evolving under (18) remain in $\tilde{S}$ and converge asymptotically to the set of critical points of the primary objective function $\tilde{H}_V$ (i.e. to the set of vectors of generators' positions and weights that yield an equitable Power diagram).

Proof: The proof is similar to that of Theorem 5.4, thus we omit it in the interest of brevity. ■

C. Obtaining cells similar to regular hexagons

In many applications it is preferable to avoid long and thin subregions. For example, in applications where a mobile agent has to service demands distributed in its own subregion, the maximum travel distance is minimized when the subregion is a circle. Thus, it is of interest to have subregions whose shapes show circular symmetry.

Define, now, the distortion functional $L_V : (A \times \mathbb{R})^m \setminus \Gamma_{\text{coinc}} \mapsto \mathbb{R}_{\geq 0}$: $\sum_{i=1}^m \int_{V_i} \|x - g_i\|^2 \lambda(x) dx$. In [12] it is shown that, when m is large, for the centroidal Voronoi diagram (i.e. centroidal Power Diagram with equal weights) that minimizes L_V, all cells are approximately congruent to a *regular hexagon*, i.e., to a polygon with considerable circular symmetry (see Section VII for a more in-depth discussion about circular symmetry).

Indeed, it is possible to obtain a Power diagram that is *close* to a centroidal Voronoi Diagram by combining control laws (16) and (18). In particular, we set up the following (distributed) control law:

$$\begin{aligned}\dot{w}_i &= u_i^{\text{cent,w}} + u_i^{\text{vor,w}},\\ \dot{g}_i &= u_i^{\text{cent,g}} + u_i^{\text{vor,g}}.\end{aligned} \tag{19}$$

Combining the results of Theorem 6.2 and Theorem 6.3, we argue that with control law (19) it is possible to obtain equitable partitions with cells *close* to regular hexagons.

VII. Simulations and Discussion

In this section we verify, through simulations, how effective is the optimization of the secondary objectives. Due to space constraint, we discuss only control law (19). We introduce two criteria to judge, respectively, *closeness* to Voronoi Diagram, and circular symmetry of a polygon.

A. Closeness to Voronoi Diagrams

In a Voronoi diagram, the intersection between the bisector of two neighboring generators g_i and g_j and the line segment joining g_i and g_j is the midpoint $g_{ij}^{\text{vor}} \doteq (g_i + g_j)/2$. Then, if we define g_{ij}^{pow} as the intersection, in a Power diagram, between the bisector of two neighboring generators g_i and g_j and the line segment joining g_i and g_j, a possible way to measure the *distance* η of a Power diagram from a Voronoi diagram is the following:

$$\eta \doteq \frac{1}{2N}\sum_{i=1}^m \sum_{j\in N_i} \frac{\|g_{ij}^{\text{pow}} - g_{ij}^{\text{vor}}\|}{0.5\gamma_{ij}} \tag{20}$$

where N is the number of neighboring relationships and, as before, $\gamma_{ij} = \|g_j - g_i\|$. Clearly, if the Power diagram is also a Voronoi diagram (i.e. if all weights are equal), we have $\eta = 0$. We will also refer to η as the Voronoi defect of a Power diagram.

B. Closeness to Regular Hexagons

A quantitative manifestation of circular symmetry is the classical *isoperimetric inequality* which says that among all planar objects of a given perimeter, the circle encloses the largest area. More precisely, given a plane region V with perimeter p_V and area $|V|$, then $p_V^2 - 4\pi|V| \geq 0$, and equality holds if and only if V is a circle. Then, we can define the *isoperimetric quotient* as follows $Q_V = \frac{4\pi|V|}{p_V^2}$; by definition, $Q_V \leq 1$, with equality only in the case of the circle. Interestingly, for a regular n-gon the isoperimetric quotient Q_n is $Q_n = \frac{\pi}{n\tan\frac{\pi}{n}}$, which converges to 1 for $n \to \infty$. Accordingly, given a partition $\mathcal{A} = \{A_i\}_{i=1}^m$, we define its isoperimetric quotient $Q_{\mathcal{A}}$ as $Q_{\mathcal{A}} \doteq \frac{1}{m}\sum Q_{A_i}$.

C. Simulation results

We are now in position to study the performance of control law (19), whose purpose is to provide equitable partitions that are *close* to Voronoi Diagrams and in which cells have high circular symmetry.

In all simulations we assume that 10 agents provide service in the unit square A. Agents' initial positions are independently and uniformly distributed over A; the initial

TABLE I

PERFORMANCE OF CONTROL LAW (19).

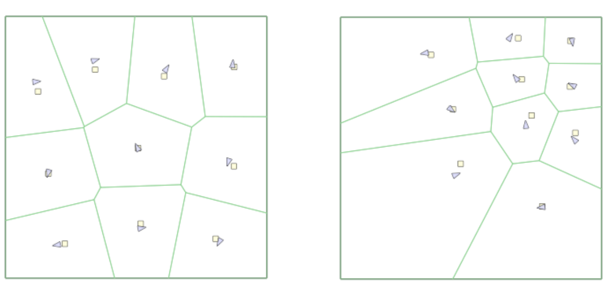
λ	$\mathbb{E}[\epsilon]$	$\max \epsilon$	$\mathbb{E}[\eta]$	$\max \eta$	$\mathbb{E}[Q]$	$\min Q$
unif	$3.8\,10^{-4}$	0.016	0.01	0.03	0.73	0.66
gauss	$3\,10^{-3}$	$5.3\,10^{-3}$	0.02	0.04	0.75	0.69

position of each virtual generator coincides with the initial position of the corresponding agent, and all weights are initialized to zero. Time is discretized with a step $dt = 0.01$, and each simulation run consists of 800 iterations (thus, the final time is $T = 8$). Define the area error ϵ as the difference, at $T = 8$, between the measure of the region of dominance with maximum measure and the measure of the region of dominance with minimum measure.

First, we consider a measure λ *uniform* over A, in particular $\lambda \equiv 1$. Therefore, we have $\lambda_A = 1$ and vehicles should reach a partition in which each region of dominance has measure equal to 0.1. For this case, we run 50 simulations.

Then, we considered a measure λ that follows a gaussian distribution, namely $\lambda(x, y) = e^{-5((x-0.8)^2+(y-0.8)^2)}$, whose peak is at the north-east corner of the unit square. Therefore, we have $\lambda_A \approx 0.336$, and vehicles should reach a partition in which each region of dominance has measure equal to 0.0336. For this case, we run 20 simulations.

Table I summarizes simulation results for the uniform λ (λ=unif) and the gaussian λ (λ=gaussian). Expectation and worst case values of, respectively, area error ϵ, Voronoi error η and isoperimetric quotient $Q_{\mathcal{A}}$ are with respect to, respectively, 50 and 20 runs. Notice that for both measures, after 800 iterations, (i) the worst case area error is within 16% from the desired measure of dominance regions, (ii) the worst case η is very close to 0, and, finally, (iii) cells have, approximately, the circular symmetry of squares (since $Q_4 \approx 0.78$). Therefore, convergence to a convex equitable partition with the desired properties (i.e., closeness to Voronoi Diagrams and circular symmetry) seems to be robust. Figure 3 shows the typical equitable partitions that are achieved with control law (19).



(a) Typical equipartition of A for $\phi(x, y) = 1$.

(b) Typical equipartition of A for $\phi(x, y) = e^{-5((x-0.8)^2+(y-0.8)^2)}$.

Fig. 3. Typical equipartitions achieved by using control law (19). Yellow dots are the virtual generators, while the gray triangles are the centroids. Notice how each bisector intersects the line segment joining the two corresponding power neighbors almost at the midpoint; hence both partitions are very close to Voronoi partitions.

VIII. CONCLUSION

In this paper an algorithm converging globally and provably to an equitable partition of the workspace is presented, together with extensions that take into account the desire to reach an equitable partition where the cells are as regular as possible. Possible applications have been rapidly sketched, and simulations results of highly non trivial configurations have been discussed showing that the algorithm and its generalizations provide global convergence to an equitable partition, while at the same time locally minimizing the distortion of the cells, according to several measures of distortion. Possible future investigations will include the extension of this set-up to the case of non-holonomic agents' dynamics, the development of new applications and, on the theoretical side, the discovery of novel properties of Voronoi and Power diagrams.

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