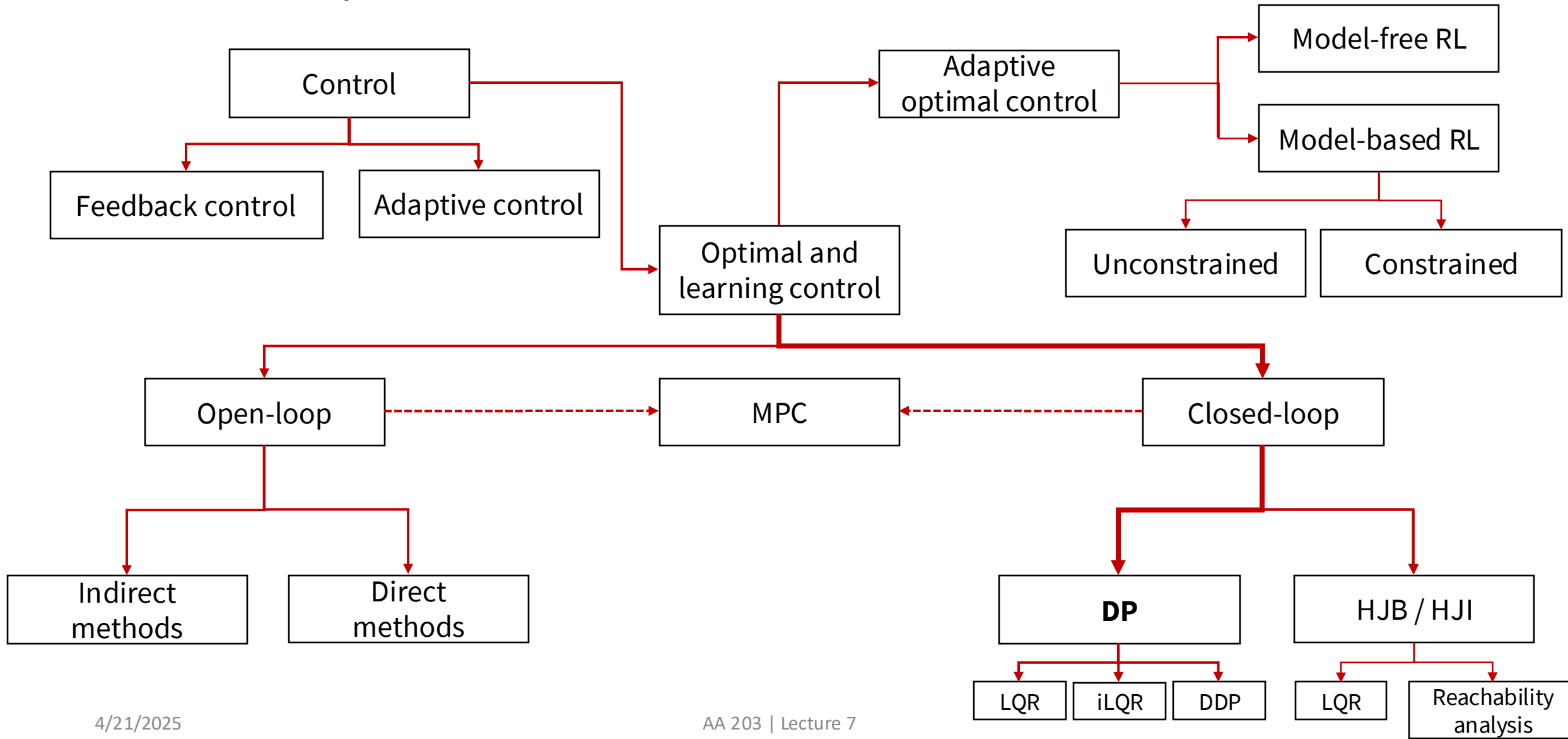


AA203

Optimal and Learning-based Control

Dynamic programming, discrete LQR

Roadmap



Basic problem – discrete-time setting

- **System:** $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k), \quad k = 0, \dots, N - 1$
- **Control constraints:** $\mathbf{u}_k \in U(\mathbf{x}_k)$
- **Cost:**

$$J(\mathbf{x}_0; \mathbf{u}_0, \dots, \mathbf{u}_{N-1}) = h_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g(\mathbf{x}_k, \mathbf{u}_k, k)$$

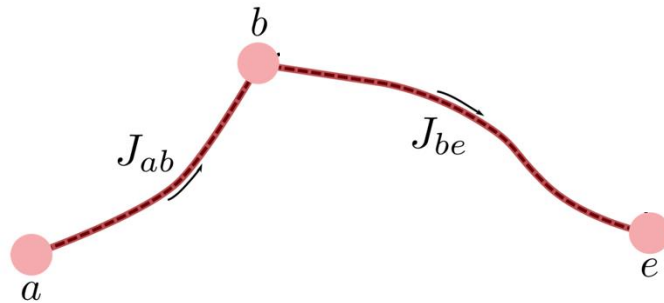
- Focus is now on finding optimal **closed-loop policies:**

$$\mathbf{u}_k^* = \pi^*(\mathbf{x}_k, k) \text{ (or } \pi_k^*(\mathbf{x}_k))$$

Principle of optimality

The **key concept** behind the dynamic programming approach is the **principle of optimality**

Suppose optimal path for a multi-stage decision-making problem is



- first decision yields segment $a - b$ with cost J_{ab}
- remaining decisions yield segments $b - e$ with cost J_{be}
- optimal cost is then $J_{ae}^* = J_{ab} + J_{be}$

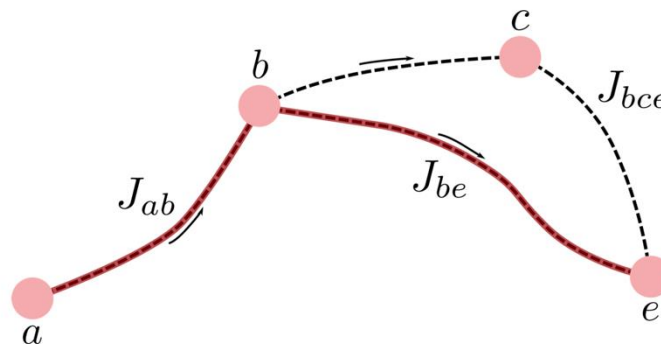
Principle of optimality

- Claim: If $a - b - e$ is optimal path from a to e , then $b - e$ is optimal path from b to e
- *Proof:* Suppose $b - c - e$ is the optimal path from b to e . Then

$$J_{bce} < J_{be}$$

and

$$J_{ab} + J_{bce} < J_{ab} + J_{be} = J_{ae}^*$$



Contradiction!

Principle of optimality

Principle of optimality: Let $\{\mathbf{u}_0^*, \mathbf{u}_1^*, \dots, \mathbf{u}_{N-1}^*\}$ be an optimal control sequence, which together with \mathbf{x}_0^* determines the corresponding state sequence $\{\mathbf{x}_0^*, \mathbf{x}_1^*, \dots, \mathbf{x}_N^*\}$. Consider the subproblem whereby we are at \mathbf{x}_k^* at time k and we wish to minimize the cost-to-go from time k to time N , i. e.,

$$g_k(\mathbf{x}_k^*, \mathbf{u}_k) + \sum_{m=k+1}^{N-1} g_m(\mathbf{x}_m, \mathbf{u}_m) + h_N(\mathbf{x}_N)$$

Then the truncated optimal sequence $\{\mathbf{u}_k^*, \mathbf{u}_{k+1}^*, \dots, \mathbf{u}_{N-1}^*\}$ is optimal for the subproblem

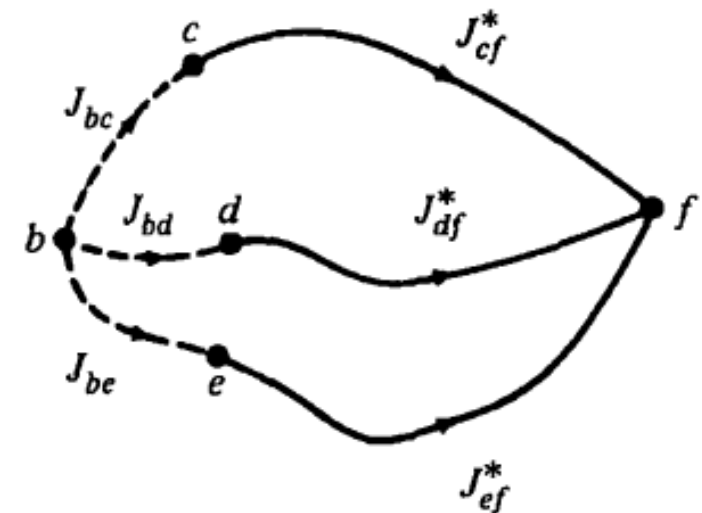
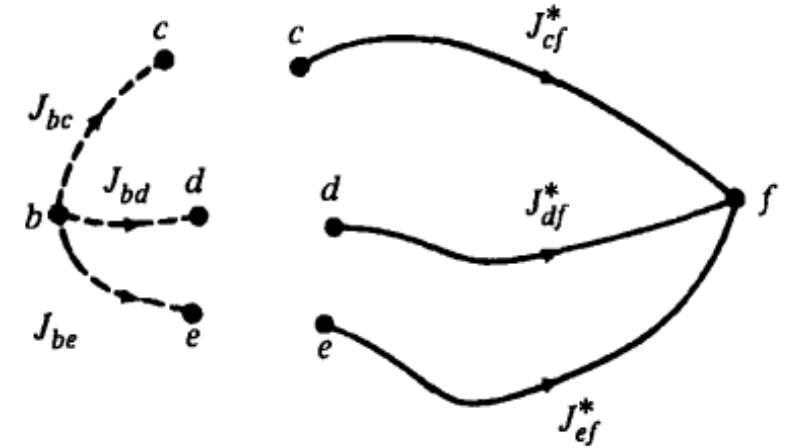
- **Tail** of optimal sequences optimal for **tail** subproblems

Applying the principle of optimality

Principle of optimality: if $b - c$ is the initial segment of the optimal path from b to f , then $c - f$ is the terminal segment of this path

Hence, the optimal trajectory is found by comparing:

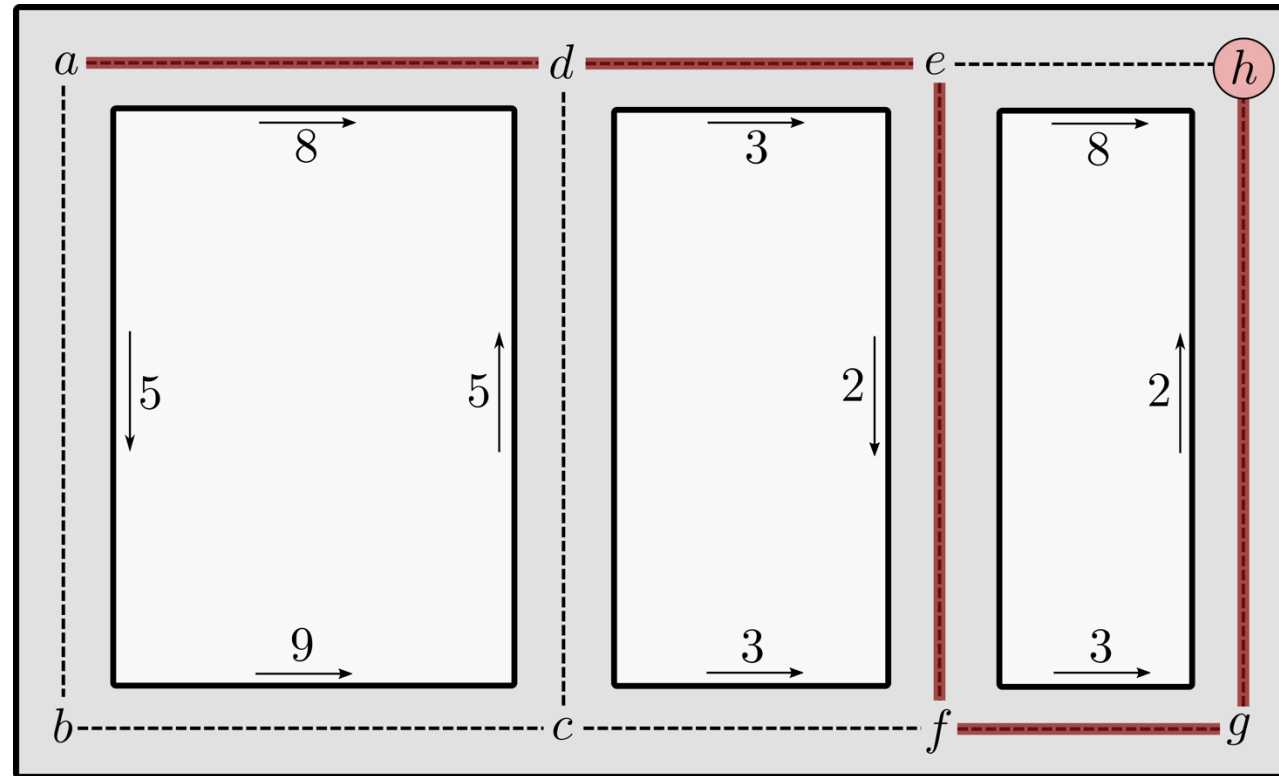
$$\begin{aligned}C_{bcf} &= J_{bc} + J_{cf}^* \\C_{bdf} &= J_{bd} + J_{df}^* \\C_{bef} &= J_{be} + J_{ef}^*\end{aligned}$$



Applying the principle of optimality

- need only to compare the concatenations of immediate decisions and optimal decisions → significant decrease in computation / possibilities
- in practice: carry out this procedure **backward** in time

Example



Optimal cost: 18

Optimal path: $a \rightarrow d \rightarrow e \rightarrow f \rightarrow g \rightarrow h$

DP Algorithm

- Model: $\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, k), \quad \mathbf{u}_k \in U(\mathbf{x}_k)$
- Cost: $J(\mathbf{x}_0) = h_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g(\mathbf{x}_k, \pi_k(\mathbf{x}_k), k)$

DP Algorithm: For every initial state \mathbf{x}_0 , the optimal cost $J^*(\mathbf{x}_0)$ is equal to $J_0(\mathbf{x}_0)$, given by the last step of the following algorithm, which proceeds backward in time from stage $N - 1$ to stage 0:

$$J_N(\mathbf{x}_N) = h_N(\mathbf{x}_N)$$

$$J_k(\mathbf{x}_k) = \min_{\mathbf{u}_k \in U(\mathbf{x}_k)} g(\mathbf{x}_k, \mathbf{u}_k, k) + J_{k+1}(f(\mathbf{x}_k, \mathbf{u}_k, k)), \quad k = 0, \dots, N - 1$$

Furthermore, if $\mathbf{u}_k^* = \pi_k^*(\mathbf{x}_k)$ minimizes the right-hand side of the above equation for each \mathbf{x}_k and k , the policy $\{\pi_0^*, \pi_1^*, \dots, \pi_{N-1}^*\}$ is optimal

Comments

- discretization (from differential equations to difference equations)
- quantization (from continuous to discrete state variables / controls)
- global minimum
- constraints, in general, simplify the numerical procedure
- optimal control in **closed-loop** form
- curse of dimensionality

Example: discrete LQR

- In most cases, DP algorithm needs to be performed numerically
- A few cases can be solved analytically

Discrete LQR: select control inputs to minimize

$$J(\mathbf{x}_0) = \frac{1}{2} \mathbf{x}'_N H \mathbf{x}_N + \frac{1}{2} \sum_{k=0}^{N-1} [\mathbf{x}'_k Q \mathbf{x}_k + \mathbf{u}'_k R \mathbf{u}_k]$$

subject to the dynamics

$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k + B_k \mathbf{u}_k$$

Assumption: $H = H' \geq 0$, $Q = Q' \geq 0$, $R = R' > 0$

Example: discrete LQR

First step:

$$J_N^*(\mathbf{x}_N) = \frac{1}{2} \mathbf{x}_N' H \mathbf{x}_N := \frac{1}{2} \mathbf{x}_N' P_N \mathbf{x}_N$$

Going backward

$$\begin{aligned} J_{N-1}(\mathbf{x}_{N-1}) = \min_{\mathbf{u}_{N-1}} & \frac{1}{2} \left\{ \mathbf{x}_{N-1}' Q \mathbf{x}_{N-1} + \mathbf{u}_{N-1}' R \mathbf{u}_{N-1} + \mathbf{x}_N' H \mathbf{x}_N \right\} \\ & \min_{\mathbf{u}_{N-1}} \frac{1}{2} \left\{ \mathbf{x}_{N-1}' Q \mathbf{x}_{N-1} + \mathbf{u}_{N-1}' R \mathbf{u}_{N-1} + \right. \\ & \left. (A_{N-1} \mathbf{x}_{N-1} + B_{N-1} \mathbf{u}_{N-1})' H (A_{N-1} \mathbf{x}_{N-1} + B_{N-1} \mathbf{u}_{N-1}) \right\} \end{aligned}$$

Example: discrete LQR

Taking derivative

$$\frac{\partial J_{N-1}^*(\mathbf{x}_{N-1})}{\partial \mathbf{u}_{N-1}} = R\mathbf{u}_{N-1} + B'_{N-1}H(A_{N-1}\mathbf{x}_{N-1} + B_{N-1}\mathbf{u}_{N-1}) = 0$$

and

$$\frac{\partial^2 J_{N-1}^*(\mathbf{x}_{N-1})}{\partial \mathbf{u}_{N-1}^2} = R + B'_{N-1}HB_{N-1} > 0$$

DP for discrete LQR

Hence, the optimizer satisfies

$$(R + B'_{N-1}HB_{N-1})\mathbf{u}^*_{N-1} + B'_{N-1}HA_{N-1}\mathbf{x}_{N-1} = 0$$

so

$$\mathbf{u}^*_{N-1} = -(R + B'_{N-1}HB_{N-1})^{-1}B'_{N-1}HA_{N-1}\mathbf{x}_{N-1} := F_{N-1}\mathbf{x}_{N-1}$$

DP for discrete LQR

Plugging in

$$\begin{aligned} J_{N-1}(\mathbf{x}_{N-1}) &= \frac{1}{2} \mathbf{x}'_{N-1} \left\{ Q + F'_{N-1} R F_{N-1} + \right. \\ &\quad \left. (A_{N-1} + B_{N-1} F_{N-1})' H (A_{N-1} + B_{N-1} F_{N-1}) \right\} \mathbf{x}_{N-1} \\ &:= \mathbf{x}'_{N-1} P_{N-1} \mathbf{x}_{N-1} \\ F_{N-1} &= - (R + B'_{N-1} P_N B_{N-1})^{-1} B'_{N-1} P_N A_{N-1} \end{aligned}$$

DP for discrete LQR

Proceeding by induction, the solution is given by

1. $J_N(\mathbf{x}_N) = \frac{1}{2} \mathbf{x}_N' P_N \mathbf{x}_N$, where $P_N = H$

2. $\mathbf{u}_k^* = F_k \mathbf{x}_k$, where $F_k = -(R + B_k' P_{k+1} B_k)^{-1} B_k' P_{k+1} A_k$

3. $J_k(\mathbf{x}_k) = \frac{1}{2} \mathbf{x}_k' P_k \mathbf{x}_k$, where

$$P_k = Q + F_k' R F_k + (A_k + B_k F_k)' P_{k+1} (A_k + B_k F_k)$$

At the end, $J_0(\mathbf{x}_0) = \frac{1}{2} \mathbf{x}_0' P_0 \mathbf{x}_0$

Next time

- Nonlinear LQR for tracking and trajectory generation