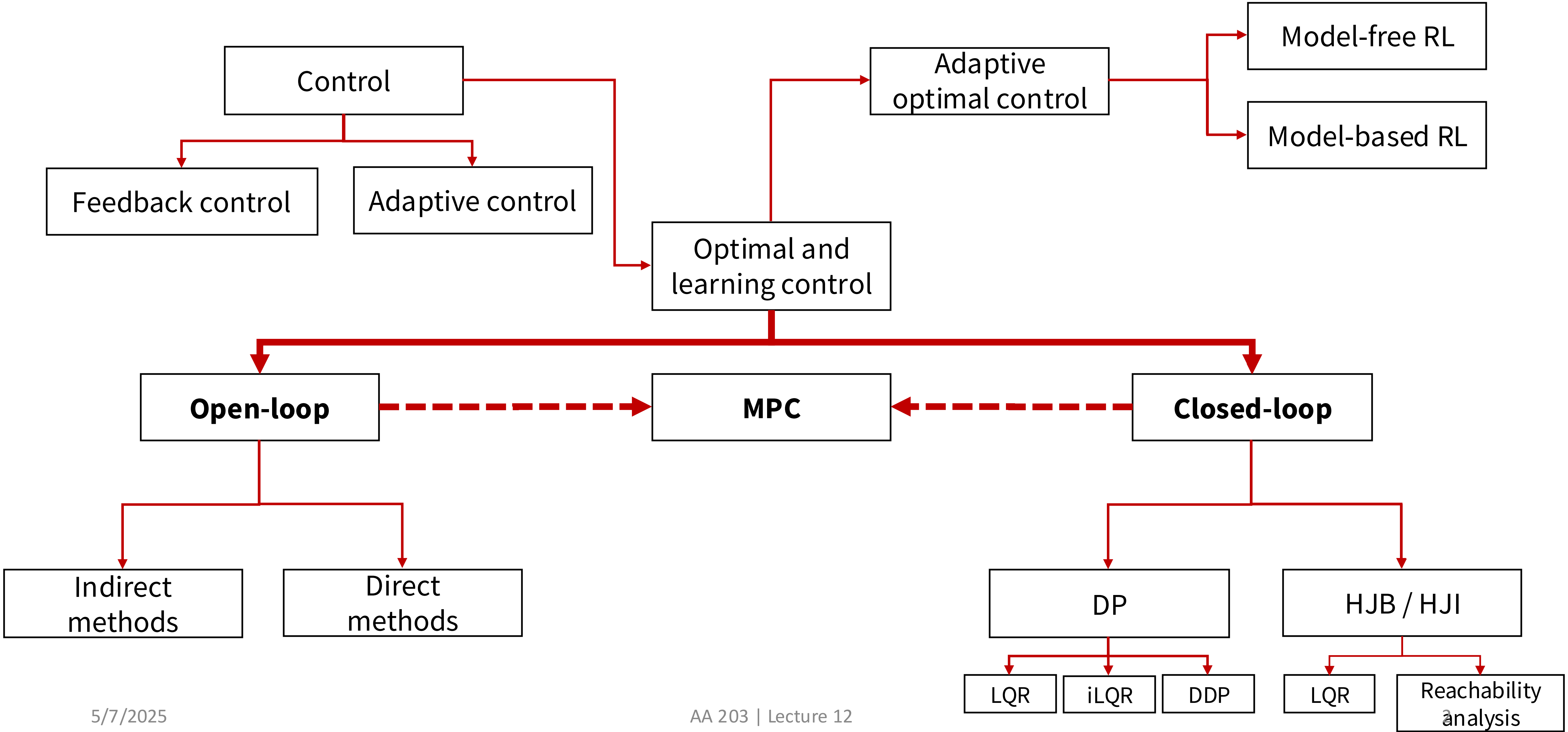
AA203
Optimal and Learning-based Control

Persistent feasibility of MPC (cont'd), stability of MPC, and explicit MPC

Stanford University

Autonomous Systems Lab

Roadmap



Control
Feedback control
Adaptive control
Optimal and learning control
Adaptive optimal control
Model-free RL
Model-based RL
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI
LQR
iLQR
DDP
LQR
Reachability analysis

Model predictive control

- Persistent feasibility of MPC (cont'd)
- Stability of MPC
- Explicit MPC

- Reading:
  - F. Borrelli, A. Bemporad, M. Morari. *Predictive Control for Linear and Hybrid Systems*, 2017.
  - J. B. Rawlings, D. Q. Mayne, M. M. Diehl. *Model Predictive Control: Theory, Computation, and Design*, 2017.

Persistent feasibility theorem

- Feasibility theorem: if set X_f is a *control invariant set* for system:

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{x}(t) \in X, \ \ \mathbf{u}(t) \in U, \ \ t \geq 0$$

then the MPC law is persistently feasible

Persistent feasibility theorem

- Proof

1. Define "truncated" feasibility set at step $N-1$:

$$X_{N-1} \coloneqq \{\mathbf{x}_{N-1} \in X \mid \exists\ \mathbf{u}_{N-1} \text{ such that } \mathbf{x}_{N-1} \in X, \mathbf{u}_{N-1} \in U,$$
$$\mathbf{x}_N \in X_f \text{ where } \mathbf{x}_N = A\mathbf{x}_{N-1} + B\mathbf{u}_{N-1}\}$$

2. Due to the terminal constraint

$$A\mathbf{x}_{N-1} + B\mathbf{u}_{N-1} = \mathbf{x}_N \in X_f$$

Persistent feasibility theorem

- Proof

3. Since X_f is a control invariant set, there exists a $\mathbf{u} \in U$ such that
$$\mathbf{x}^+ = A\mathbf{x}_N + B\mathbf{u} \in X_f$$
4. The above is indeed the requirement to belong to set X_{N-1}
5. Thus, $A\mathbf{x}_{N-1} + B\mathbf{u}_{N-1} = \mathbf{x}_N \in X_{N-1}$
6. We have just proved that X_{N-1} is control invariant
7. Repeating this argument, one can recursively show that $X_{N-2}, X_{N-3}, \cdots, X_1$ are control invariant, and the persistent feasibility lemma then applies

Practical aspects of persistent feasibility

- The terminal set X_f is introduced *artificially* for the sole purpose of leading to a *sufficient condition* for persistent feasibility
- We want it to be large so that it does not compromise closed-loop performance
- Though it is simplest to choose $X_f = \{0\}$, this is generally undesirable
- We'll discuss better choices later

Stability of MPC

- Persistent feasibility does not guarantee that the closed-loop trajectories converge towards the desired equilibrium point

- One of the most popular approaches to guarantee persistent feasibility and stability of the MPC law makes use of a control invariant terminal set X_f for feasibility, and of a terminal function $p(\cdot)$ for stability

- To prove stability, we leverage the tool of Lyapunov stability theory

Lyapunov stability theory

- Lyapunov theorem: Consider the equilibrium point $\mathbf{x} = 0$ for the autonomous system $\{\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)\}$ (with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$). Let $\Omega \subset \mathbb{R}^n$ be a closed, bounded, positively invariant set containing the origin. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a function, continuous at the origin, such that

$$V(\mathbf{0}) = 0 \text{ and } V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \Omega \setminus \{\mathbf{0}\}$$

$$V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) < 0 \quad \forall \mathbf{x}_k \in \Omega \setminus \{\mathbf{0}\}$$

  Then $\mathbf{x} = 0$ is asymptotically stable in Ω.

- The idea is to show that with appropriate choices of X_f and $p(\cdot)$, J_0^* is a Lyapunov function for the closed-loop system

MPC stability theorem

- MPC stability theorem (for quadratic cost): Assume

A0: $Q = Q^T \succ 0, R = R^T \succ 0, P \succ 0$

A1: Sets X, X_f, and U contain the origin in their interior and are closed

A2: $X_f \subseteq X$ is control invariant and bounded

A3: $\min_{\mathbf{u} \in U,\, A\mathbf{x}+B\mathbf{u} \in X_f} \big(-p(\mathbf{x}) + c(\mathbf{x}, \mathbf{u}) + p(A\mathbf{x} + B\mathbf{u})\big) \leq 0, \forall \mathbf{x} \in X_f$

Then, the origin of the closed-loop system is asymptotically stable with domain of attraction X_0.

MPC stability theorem

- Proof:

1. Note that, by assumption A2, persistent feasibility is guaranteed for *any* P, Q, R
2. We want to show that J_0^* is a Lyapunov function for the closed-loop system $\mathbf{x}(t+1) = \mathbf{f}_{\text{cl}}(\mathbf{x}(t))$, with respect to the equilibrium $\mathbf{f}_{\text{cl}}(\mathbf{0}) = \mathbf{0}$ (the origin is indeed an equilibrium as $\mathbf{0} \in X, \mathbf{0} \in U$, and the cost is positive for any non-zero control sequence)
3. X_0 is bounded and closed (follows from assumption on X_f)
4. $J_0^*(\mathbf{0}) = 0$ (value is nonnegative by construction, and 0 is achievable)

MPC stability theorem

- Proof:

5. $J_0^*(\mathbf{x}) > 0$ for all $\mathbf{x} \in X_0 \setminus \{\mathbf{0}\}$
6. Next we show the decay property. Since the setup is time-invariant, we can study the decay property between $t = 0$ and $t = 1$
   - Let $\mathbf{x}(0) \in X_0$, let $U_0^{[0]} = [\mathbf{u}_0^{[0]}, \mathbf{u}_1^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}]$ be the optimal control sequence, and let $[\mathbf{x}(0), \mathbf{x}_1^{[0]}, \dots, \mathbf{x}_N^{[0]}]$ be the corresponding trajectory
   - After applying $\mathbf{u}_0^{[0]}$, one obtains $\mathbf{x}(1) = A\mathbf{x}(0) + B\mathbf{u}_0^{[0]}$
   - Consider the sequence of controls $[\mathbf{u}_1^{[0]}, \mathbf{u}_2^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}, \mathbf{v}]$, where $\mathbf{v} \in U$, and the corresponding state trajectory is $[\mathbf{x}(1), \mathbf{x}_2^{[0]}, \dots, \mathbf{x}_N^{[0]}, A\mathbf{x}_N^{[0]} + B\mathbf{v}]$

MPC stability theorem

- Since $\mathbf{x}_N^{[0]} \in X_f$ (by terminal constraint), and since X_f is control invariant,
$$\exists \bar{\mathbf{v}} \in U \text{ such that } A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}} \in X_f$$
- With such a choice of $\bar{\mathbf{v}}$, the sequence $[\mathbf{u}_1^{[0]}, \mathbf{u}_2^{[0]}, \ldots, \mathbf{u}_{N-1}^{[0]}, \bar{\mathbf{v}}]$ is feasible for the MPC optimization problem at time $t = 1$
- Since this sequence is not necessarily optimal

$$J_0^*(\mathbf{x}(1)) \leq p\left(A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}}\right) + \sum_{k=1}^{N-1} c\left(\mathbf{x}_k^{[0]}, \mathbf{u}_k^{[0]}\right) + c\left(\mathbf{x}_N^{[0]}, \bar{\mathbf{v}}\right)$$

MPC stability theorem

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$$+ p\left(\mathbf{x}_N^{[0]}\right) - p\left(\mathbf{x}_N^{[0]}\right) + c\left(\mathbf{x}(0), \mathbf{u}_0^{[0]}\right) - c\left(\mathbf{x}(0), \mathbf{u}_0^{[0]}\right)$$

MPC stability theorem

- Equivalently

$$J_0^*(\mathbf{x}(1)) \le p\left(A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}}\right) + J_0^*(\mathbf{x}(0)) - p\left(\mathbf{x}_N^{[0]}\right) - c\left(\mathbf{x}(0), \mathbf{u}_0^{[0]}\right) + c(\mathbf{x}_N^{[0]}, \bar{\mathbf{v}})$$

- Since $\mathbf{x}_N^{[0]} \in X_f$, by assumption A3, we can select $\bar{\mathbf{v}}$ such that

$$J_0^*(\mathbf{x}(1)) \le J_0^*(\mathbf{x}(0)) - c\left(\mathbf{x}(0), \mathbf{u}_0^{[0]}\right)$$

- Since $c\left(\mathbf{x}(0), \mathbf{u}_0^{[0]}\right) > 0$ for all $\mathbf{x}(0) \in X_0 \setminus \{0\}$,

$$J_0^*(\mathbf{x}(1)) - J_0^*(\mathbf{x}(0)) < 0$$

- The last step is to prove continuity; details are omitted and can be found in Borrelli, Bemporad, Morari, 2017
- Note: A2 is used to guarantee persistent feasibility; this assumption can be replaced with an assumption on the horizon N

How to choose X_f and P?

- Case 1: assume A is asymptotically stable
  - Set X_f as the maximally positive invariant set O_∞ for system $\mathbf{x}(t+1) = A\mathbf{x}(t), \quad \mathbf{x}(t) \in X$
  - X_f is a control invariant set for system $\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t)$, as $\mathbf{u} = 0$ is a feasible control
  - As for stability, $\mathbf{u} = 0$ is feasible and $A\mathbf{x} \in X_f$ if $\mathbf{x} \in X_f$, thus assumption A3 becomes

$$-\mathbf{x}^T P\mathbf{x} + \mathbf{x}^T Q\mathbf{x} + \mathbf{x}^T A^T PA\mathbf{x} \leq 0, \text{ for all } \mathbf{x} \in X_f,$$

which is true since, due to the fact that A is asymptotically stable,

$$\exists P \succ 0 \mid -P + Q + A^T PA = 0$$ (Lyapunov Equation)

Cost-to-go/value function

How to choose X_f and P?

- Case 2: general case (e.g., if A is open-loop unstable)
  - Let F_∞ be the optimal gain for the infinite-horizon LQR controller
  - Set X_f as the maximal positive invariant set for system

$$\mathbf{x}(t+1) = (A + BF_\infty)\mathbf{x}(t)$$

    (with constraints $\mathbf{x}(t) \in X$, and $F_\infty \mathbf{x}(t) \in U$)
  - Set P as the solution P_∞ to the discrete-time Riccati equation, i.e., the value function via LQR

$$-P + Q + A^T PA - (A^T PB)(R + B^T PB)^{-1}(B^T PA) = 0$$

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- Note: both cases as presented are just (suboptimal) choices!

Explicit MPC

- In some cases, the MPC law can be *pre-computed* → no need for online optimization
- Important case: constrained LQR

$$J_0^*(\mathbf{x}) = \min_{\mathbf{u}_0,\ldots,\mathbf{u}_{N-1}} \mathbf{x}_N^T P \mathbf{x}_N + \sum_{k=0}^{N-1} \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k$$

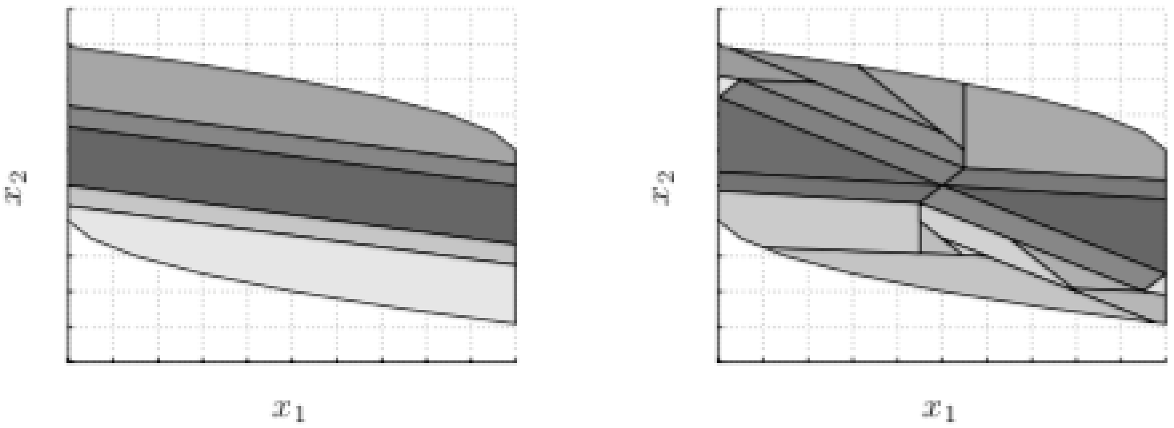
$$\begin{aligned} \text{subject to} \quad & \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \quad && k = 0, \ldots, N-1 \\ & \mathbf{x}_k \in X, \quad \mathbf{u}_k \in U, \quad && k = 0, \ldots, N-1 \\ & \mathbf{x}_N \in X_f \\ & \mathbf{x}_0 = \mathbf{x} \end{aligned}$$

Explicit MPC

- The solution to the constrained LQR problem is a control which is a continuous piecewise affine function on polyhedral partition of the state space X, that is $\mathbf{u}_k^* = \pi_k(\mathbf{x}_k)$ where

$$\pi_k(\mathbf{x}) = F_k^j \mathbf{x} + g_k^j \text{ if } H_k^j \mathbf{x} \leq K_k^j, \ \ j = 1, \ldots, N_k^r$$

- Thus, online, one has to locate in which cell of the polyhedral partition the state $\mathbf{x}$ lies, and then one obtains the optimal control via a look-up table query



Tuning and practical use

- At present there is no other technique other than MPC to design controllers for general large linear multivariable systems with input and output constraints with a stability guarantee
- Design approach (for squared 2-norm cost):
  - Choose horizon length N and the control invariant target set X_f
  - Control invariant target set X_f should be as large as possible for performance
  - Choose the parameters Q and R freely to affect the control performance
  - Adjust P as per the stability theorem
  - Useful toolbox (MATLAB): https://www.mpt3.org/
- In practice, sometimes choosing a good terminal cost is enough (i.e., don't need to enforce a terminal control invariant condition), though you may be sacrificing guarantees

MPC for reference tracking

- Usual cost

$$\sum_{k=0}^{N-1} \mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k$$

  does not work, as in steady state control does not need to be zero
- $\delta\mathbf{u}$- formulation: reason in terms of *control changes*

$$\mathbf{u}_k = \mathbf{u}_{k-1} + \delta\mathbf{u}_k$$

MPC for reference tracking

- The MPC problem is readily modified to

$$J_0^*(\mathbf{x}(t)) = \min_{\delta\mathbf{u}_0,\ldots,\delta\mathbf{u}_{N-1}} \sum_k \|\mathbf{y}_k - \mathbf{r}_k\|_Q^2 + \|\delta\mathbf{u}_k\|_R^2$$

$$\begin{aligned}\text{subject to}\quad & \mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, && k = 0,\ldots,N-1\\ & \mathbf{y}_k = C\mathbf{x}_k, && k = 0,\ldots,N-1\\ & \mathbf{x}_k \in X, \quad \mathbf{u}_k \in U, && k = 0,\ldots,N-1\\ & \mathbf{x}_N \in X_f && \\ & \mathbf{u}_k = \mathbf{u}_{k-1} + \delta\mathbf{u}_k, && k = 0,\ldots,N-1\\ & \mathbf{x}_0 = \mathbf{x}(t), \quad \mathbf{u}_{-1} = \mathbf{u}(t-1) && \end{aligned}$$

- The control input is then $\mathbf{u}(t) = \delta\mathbf{u}_0^* + \mathbf{u}(t-1)$

Next time

- Intro to learning
- Sys ID
- Adaptive control