Stanford **Spring 2025**

AA 203: Optimal and Learning-based Control
Homework #1
Due April 21 by 11:59 pm

Learning goals for this problem set:

Problem 1: To gain insights into the implementation of gradient methods and review some notions of linear algebra.

Problem 2: To familiarize with Linear Quadratic control and learn a first algorithmic approach to this problem.

Problem 3: Become familiar with the process of solving calculus of variations problems.

Problem 4: To familiarize with the Hamiltonian equations for optimal control.

1.1 Gradient descent and line search. Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric positive-definite matrix, and $b \in \mathbb{R}^n$ be a given vector. Consider the quadratic optimization problem

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}x^\mathsf{T}Qx - b^\mathsf{T}x.$$

Let $f(x) \coloneqq \frac{1}{2}x^\mathsf{T}Qx - b^\mathsf{T}x$, and denote the eigenvalues of Q as $\lambda_1, \dots, \lambda_n$.

(a) Find the unique local minimum candidate $x^* \in \mathbb{R}^n$. Prove x^* is a global minimum.

Hint: Any twice-differentiable function f is strictly convex if the Hessian $\nabla^2 f(x)$ is positive-definite for all $x \in \mathbb{R}^n$.

(b) Show that, starting from any initial point $x^{(0)} \in \mathbb{R}^n$, Newton's method with constant step size $\eta = 1$ converges in one iteration to the optimal solution x^*. Hence, performing one step of Newton's method is equivalent to solving the linear system of equations $Qx = b$. What would be the downside of this solution method if n is large (e.g., $n \gg 10^4$) and the matrix Q has no particular structure?

(c) Let $S \in \mathbb{R}^{n\times n}$ be a symmetric matrix. By the Spectral Theorem, there exist an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma = \operatorname{diag}(\mu_1, \dots, \mu_n)$ such that $S = U\Sigma U^\mathsf{T}$. Show $\|Sx\|_2 = \|\Sigma U^\mathsf{T}x\|_2$ for any $x \in \mathbb{R}^n$. Then show $\|\Sigma z\|_2 \le \max_{i\in\{1,\dots,n\}}|\mu_i|\|z\|_2$ for any $z \in \mathbb{R}^n$. Finally, conclude that $\|Sx\|_2 \le \max_{i\in\{1,\dots,n\}}|\mu_i|\|x\|_2$ for any $x \in \mathbb{R}^n$.

Hint: If $U \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then $\|Uy\|_2 = \|U^\mathsf{T}y\|_2 = \|y\|_2$ for any $y \in \mathbb{R}^n$.

(d) For any $\eta > 0$, show that the eigenvalues of the matrix $I - \eta Q$ are exactly $\{1 - \eta\lambda_i\}_{i=1}^n$.

Hint: Identify an orthonormal basis of vectors $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$ such that $(I - \eta Q)v_i = (1 - \eta\lambda_i)v_i$ for each i.

(e) Consider the gradient descent update rule $x^{(k+1)} = x^{(k)} - \eta\,\nabla f(x^{(k)})$ at iteration $k \in \mathbb{N}_{\ge 0}$ with a constant step size $\eta > 0$. Define $\delta^{(k)} \coloneqq \|x^{(k)} - x^*\|_2$ and $\gamma(\eta) \coloneqq \max_{i\in\{1,\dots,n\}}|1 - \eta\lambda_i|$. Use an inductive argument to show $\delta^{(k)} \le \gamma(\eta)^k\delta_0$ for all $k \in \mathbb{N}_{\ge 0}$.

(f) Consider gradient descent with exact line search. At each iteration k, denote the descent direction by $d^{(k)} := -\nabla f(x^{(k)})$ and the optimal step size by

$$\eta^{(k)} := \arg\min_{\eta \geq 0} f(x^{(k)} + \eta d^{(k)}).$$

Prove

$$\eta^{(k)} = \frac{\|d^{(k)}\|_2^2}{{d^{(k)}}^\top Q d^{(k)}}.$$

(g) For $n = 2$ and $f(x) = \frac{1}{2}(x_1^2 + \gamma x_2^2)$ with $\gamma = 10$, what is the optimal solution x^*? Implement gradient descent with a constant step size and exact line search, starting from $x^{(0)} = (5, 1)$ and $x^{(0)} = (1, 5)$. What do you observe with exact line search? When does gradient descent begin to "zig-zag"? What issue do you observe with a constant step size? Repeat both experiments with $\gamma = 1$. Submit your plots.

1.2 LQR as a QP. Consider the Linear Time-Invariant (LTI) dynamical system

$$x_{t+1} = Ax_t + Bu_t,$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are given matrices, and $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ are the system state and applied control input, respectively, at time $t \in \mathbb{N}_{\geq 0}$.

Let $x_0 \in \mathbb{R}^n$ be the fixed initial state and $T \in \mathbb{N}$ be some time horizon. Our goal is to find a sequence of control inputs $u^* := (u_0^*, u_1^*, \dots, u_{T-1}^*) \in \mathbb{R}^{mT}$ that minimizes the quadratic cost

$$J(u) := x_T^\mathsf{T} Q_T x_T + \sum_{t=0}^{T-1} \Big(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t \Big),$$

where $Q_T \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times n}$, and $R \in \mathbb{R}^{m \times m}$ are positive-definite matrices. Later, we will see how dynamic programming can be used to derive an elegant, recursive solution to this problem. For now, we study a convex least-squares formulation. Specifically, we reformulate the problem of minimizing $J(u)$ as

$$\min_{u \in \mathbb{R}^{mT}} \frac{1}{2} u^\mathsf{T} \tilde{Q} u - \tilde{b}^\mathsf{T} u,$$

where $u := (u_0, u_1, \dots, u_{T-1}) \in \mathbb{R}^{mT}$ is the vector of stacked control inputs, $\tilde{Q} \in \mathbb{R}^{mT \times mT}$ is a positive-definite matrix, and $\tilde{b} \in \mathbb{R}^{mT}$.

(a) Write down $\tilde{Q}$ and $\tilde{b}$ in terms of Q_T, Q, R, A, B, and x_0.

(b) With this reformulation, implement the gradient descent algorithm of your choice to compute the optimal sequence of control inputs u^* for

$$Q_T = 10 I_2, \quad Q = I_2, \quad R = I_1, \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T = 20,$$

where I_n is the identity matrix with dimension n. What is the optimal cost $J(u^*)$?

1.3 Extremal curves. Given the functional

$$J(x) = \int_0^1 \left(\frac{1}{2}\dot{x}(t)^2 + 5x(t)\dot{x}(t) + x(t)^2 + 5x(t) \right) dt,$$

find an extremal curve $x^* : [0,1] \to \mathbb{R}$ that satisfies $x^*(0) = 1$ and $x^*(1) = 3$.

1.4 Zermelo's ship. Zermelo's ship must travel through a region of strong currents. The position of the ship is denoted by $(x(t), y(t)) \in \mathbb{R}^2$. The ship travels at a constant speed $v > 0$, yet its heading $\theta(t)$ can be controlled. The current moves in the positive x-direction with speed $w(y(t))$. The equations of motion for the ship are

$$\begin{aligned} \dot{x}(t) &= v \cos \theta(t) + w(y(t)) \\ \dot{y}(t) &= v \sin \theta(t) \end{aligned}.$$

We want to control the heading $\theta(t)$ such that the ship travels from a given initial position $(x(t_0), y(t_0)) = (x_0, y_0)$ to the origin $(0, 0)$ in minimum time.

(a) Suppose $w(y(t)) = \frac{v}{h} y(t)$, where $h > 0$ is a known constant. Show that an optimal control law $\theta^*(t)$ must satisfy a linear tangent law of the form

$$\tan \theta^*(t) = \alpha - \frac{v}{h} t$$

for some constant $\alpha \in \mathbb{R}$.

(b) Suppose $w(y(t)) \equiv \beta$ for some constant $\beta > 0$. Derive an expression for the optimal transfer time $t_1^* - t_0$.