AA203
Optimal and Learning-based Control

Pontryagin’s Minimum Principle (PMP);
computational methods
Roadmap

Control

Feedback control
Adaptive control

Optimal and learning control

Direct methods
Indirect methods

Open-loop
MPC
Closed-loop

Adaptive optimal control
Model-free RL
Model-based RL

Unconstrained
Constrained

Reachability analysis
LQR iLQR
DP
LQR DDP
HJB / HJI

LQR iLQR
DDP

Optimal and learning control
Adaptive control
Feedback control
Open-loop
Indirect methods
Direct methods
Outline

• Necessary conditions for optimal control with bounded controls:
  • Pontryagin’s Minimum Principle (PMP)
• Examples: Applications of PMP (and insights we can derive from the analysis)
• Computational methods
Necessary conditions for optimal control (with unbounded controls)

• The problem is to find an admissible control $u^*$ which causes the system

$$\dot{x}(t) = f(x(t), u(t), t)$$

to follow an admissible trajectory $x^*$ that minimizes the functional

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t) \, dt$$

• Assumptions: $h \in C^2$, state and control regions are unbounded, $t_0$ and $x(0)$ are fixed
Necessary conditions for optimal control
(with unbounded controls)

• Define the Hamiltonian

\[ H(x(t), u(t), p(t), t) := g(x(t), u(t), t) + p(t)^T f(x(t), u(t), t) \]

• The necessary conditions for optimality (proof to follow) are

\[
\begin{align*}
\dot{x}^*(t) &= \frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) \\
\dot{p}^*(t) &= -\frac{\partial H}{\partial x}(x^*(t), u^*(t), p^*(t), t) \\
0 &= \frac{\partial H}{\partial u}(x^*(t), u^*(t), p^*(t), t)
\end{align*}
\]

for all \( t \in [t_0, t_f] \)

with boundary conditions

\[
\left[ \frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f) \right]^T \delta x_f + \left[ H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial h}{\partial t}(x^*(t_f), t_f) \right] \delta t_f = 0
\]
Necessary conditions for optimal control (with \textit{bounded} controls)

• So far, we have assumed that the admissible controls and states are not constrained by any boundaries

• However, in realistic systems, such constraints do commonly occur
  • control constraints often occur due to actuation limits
  • state constraints often occur due to safety considerations

• We will now consider the case with control constraints, which will lead to the statement of the Pontryagin’s minimum principle
Why do control constraints complicate the analysis?

• By definition, the control $u^*$ causes the functional $J$ to have a relative minimum if

$$J(u) - J(u^*) = \Delta J \geq 0$$

for all admissible controls “close” to $u^*$

• If we let $u = u^* + \delta u$, the increment in $J$ can be expressed as

$$\Delta J(u^*, \delta u) = \delta J(u^*, \delta u) + \text{higher order terms}$$

• The variation $\delta u$ is arbitrary only if the extremal control is strictly within the boundary for all time in the interval $[t_0, t_f]$

• In general, however, an extremal control lies on a boundary during at least one subinterval of the interval $[t_0, t_f]$
Why do control constraints complicate the analysis?

- As a consequence, admissible control variations $\delta u$ exist whose negatives $(-\delta u)$ are not admissible.
- This implies that a necessary condition for $u^*$ to minimize $J$ is
  \[ \delta J(u^*, \delta u) \geq 0 \]
  for all admissible variations with $||\delta u||$ small enough.
Pontryagin’s minimum principle

• Assuming bounded controls $u \in U$, the necessary optimality conditions are ($H$ is the Hamiltonian)

\[
\dot{x}^*(t) = \frac{\partial H}{\partial p} (x^*(t), u^*(t), p^*(t), t)
\]

\[
\dot{p}^*(t) = -\frac{\partial H}{\partial x} (x^*(t), u^*(t), p^*(t), t)
\]

\[
H(x^*(t), u^*(t), p^*(t), t) \leq H(x^*(t), u(t), p^*(t), t), \text{ for all } u(t) \in U
\]

along with the boundary conditions:

\[
\left[ \frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f) \right]^T \delta x_f + \left[ H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial h}{\partial t}(x^*(t_f), t_f) \right] \delta t_f = 0
\]
Pontryagin’s minimum principle

• $u^*(t)$ is a control that causes $H(x^*(t), u(t), p^*(t), t)$ to assume its global minimum
• Harder condition in general to analyze
• Example: consider the system having dynamics:
  \[
  \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -x_2(t) + u(t);
  \]
  it is desired to minimize the functional
  \[
  J = \int_{t_0}^{t_f} \frac{1}{2} [x_1^2(t) + u^2(t)] dt
  \]
  subject to the control constraint $|u(t)| \leq 1$ with $t_f$ fixed and the final state free.
Pontryagin’s minimum principle

Solution:

• If the control is unconstrained,
  \[ u^*(t) = -p_2^*(t) \]

• If the control is constrained as \(|u(t)| \leq 1\), then
  \[
  u^*(t) = \begin{cases} 
  -1 & \text{for } 1 < p_2^*(t) \\
  -p_2^*(t), & -1 \leq p_2^*(t) \leq 1 \\
  +1 & \text{for } p_2^*(t) < -1 
  \end{cases} 
  \]

• To determine \( u^*(t) \) explicitly, the state and co-state equations must still be solved
Additional necessary conditions

1. If the final time is fixed and the Hamiltonian does not depend explicitly on time, then
   \[ H(x^*(t), u^*(t), p^*(t)) = c \text{ for all } t \in [t_0, t_f] \]

2. If the final time is free and the Hamiltonian does not depend explicitly on time, then
   \[ H(x^*(t), u^*(t), p^*(t)) = 0 \text{ for all } t \in [t_0, t_f] \]
Minimum time problems

• Find the control input sequence

\[ M_i^- \leq u_i(t) \leq M_i^+ \text{ for } i = 1, \ldots, m \]

that drives the control affine system

\[ \dot{x} = a(x, t) + B(x, t)u(t) \]

from an arbitrary state \( x_0 \) to the origin, and minimizes time

\[ J = \int_{t_0}^{t_f} 1 \, dt \]
Minimum time problems

• Form the Hamiltonian

\[ H = 1 + p(t)^T \{ a(x, t) + B(x, t)u(t) \} \]

\[ = 1 + p(t)^T \{ a(x, t) + [b_1(x, t) \ b_2(x, t) \ldots b_m(x, t)]u(t) \} \]

\[ = 1 + p(t)^T a(x, t) + \sum_{i=1}^{m} p(t)^T b_i(x, t)u_i(t) \]

• By the PMP, select \( u_i(t) \) to minimize \( H \), which gives

\[ u_i^*(t) = \begin{cases} 
M_i^+ & \text{if } p(t)^T b_i(x, t) < 0 \\
M_i^- & \text{if } p(t)^T b_i(x, t) > 0 
\end{cases} \]

“Bang-bang” control
Minimum time problems

• Note: we showed what to do when $\mathbf{p}(t)^T \mathbf{b}_i(x, t) \neq 0$

• Not obvious what to do if $\mathbf{p}(t)^T \mathbf{b}_i(x, t) = 0$

• If $\mathbf{p}(t)^T \mathbf{b}_i(x, t) = 0$ for some finite time interval, then the coefficient of $u_i(t)$ in the Hamiltonian is zero, so the PMP provides no information on how to select $u_i(t)$

• The treatment of such a singular condition requires a more sophisticated analysis

• The analysis in the linear case is significantly easier, see Kirk Sec. 5.4
Minimum fuel problems

• Find the control input sequence
  \[ M_i^- \leq u_i(t) \leq M_i^+ \text{ for } i = 1, \ldots, m \]
  that drives the control affine system
  \[ \dot{x} = a(x, t) + B(x, t)u(t) \]
  from an arbitrary state \( x_0 \) to the origin in a fixed time, and minimizes
  \[ J = \int_{t_0}^{t_f} \sum_{i=1}^{m} c_i |u_i(t)| \, dt \]
Minimum fuel problems

• Form the Hamiltonian

\[ H = \sum_{i=1}^{m} c_i \ |u_i(t)| + p(t)^T \{a(x, t) + B(x, t)u(t)\} \]

\[ = \sum_{i=1}^{m} c_i \ |u_i(t)| + p(t)^T a(x, t) + \sum_{i=1}^{m} p(t)^T b_i(x, t)u_i(t) \]

\[ = \sum_{i=1}^{m} [c_i \ |u_i(t)| + p(t)^T b_i(x, t)u_i(t)] + p(t)^T a(x, t) \]

• By the PMP, select \( u_i(t) \) to minimize \( H \), that is

\[ \sum_{i=1}^{m} [c_i \ |u_i(t)| + p(t)^T b_i(x, t)u_i(t)] \leq \sum_{i=1}^{m} [c_i \ |u_i(t)| + p(t)^T b_i(x, t)u_i(t)] \]
Minimum fuel problems

- Since the components of $u(t)$ are independent, then one can just look at
  \[ c_i |u_i^*(t)| + p(t)^T b_i(x, t)u_i^*(t) \leq c_i |u_i(t)| + p(t)^T b_i(x, t)u_i(t) \]
- The resulting control law is
  \[
  u_i^*(t) = \begin{cases} 
  M^-_i & \text{if } c_i < p(t)^T b_i(x, t) \\
  0 & \text{if } -c_i < p(t)^T b_i(x, t) < c_i \\
  M^+_i & \text{if } p(t)^T b_i(x, t) < -c_i 
  \end{cases}
  \]
  “Bang-off-bang” control
Minimum energy problems

• Find the control input sequence

\[ M_i^- \leq u_i(t) \leq M_i^+ \text{ for } i = 1, \ldots, m \]

that drives the control affine system

\[ \dot{x} = a(x, t) + B(x, t)u(t) \]

from an arbitrary state \( x_0 \) to the origin in a fixed time, and minimizes

\[ J = \frac{1}{2} \int_{t_0}^{t_f} u(t)^T R u(t) dt , \]

where \( R > 0 \) and diagonal
Minimum energy problems

• Form the Hamiltonian

\[ H = \frac{1}{2} \mathbf{u}(t)^T R \mathbf{u}(t) + \mathbf{p}(t)^T \{ \mathbf{a}(\mathbf{x}, t) + B(\mathbf{x}, t)\mathbf{u}(t) \} \]

\[ = \frac{1}{2} \mathbf{u}(t)^T R \mathbf{u}(t) + \mathbf{p}(t)^T B(\mathbf{x}, t)\mathbf{u}(t) + \mathbf{p}(t)^T \mathbf{a}(\mathbf{x}, t) \]

• By the PMP, we need to solve

\[ \mathbf{u}^*(t) = \arg \min_{\mathbf{u}(t) \in \mathcal{U}} \left[ \sum_{i=1}^{m} \frac{1}{2} R_{ii} u_i(t)^2 + \mathbf{p}(t)^T \mathbf{b}_i(\mathbf{x}, t) u_i(t) \right] \]
Minimum energy problems

• As in the first example today, in the unconstrained case, the optimal solution for each component of $\mathbf{u}(t)$ would be

$$\hat{u}_i(t) = -R_{ii}^{-1} p(t)^T \mathbf{b}_i(x, t)$$

• Considering the input constraints, the resulting control law is

$$u^*(t) = \begin{cases} M_i^- & \text{if } \hat{u}_i(t) < M_i^- \\ \hat{u}_i(t) & \text{if } M_i^- < \hat{u}_i(t) < M_i^+ \\ M_i^+ & \text{if } M_i^+ < \hat{u}_i(t) \end{cases}$$

“Saturating” control
Uniqueness and existence

• Note: uniqueness and existence are not in general guaranteed!

• Example 1 (non uniqueness): find a control sequence $u(t)$ to transfer the system $\dot{x}(t) = u(t)$ from an arbitrary initial state $x_0$ to the origin, and such that the functional $J = \int_0^{t_f} |u(t)| dt$ is minimized. The final time is free, and the admissible controls are $|u(t)| \leq 1$

• Example 2 (non existence): find a control sequence $u(t)$ to transfer the system $\dot{x}(t) = x(t) + u(t)$ from an arbitrary initial state $x_0$ to the origin, and such that the functional $J = \int_{t_0}^{t_f} |u(t)| dt$ is minimized. The final time is free, and the admissible controls are $|u(t)| \leq 1$
Computational methods

• Until now, we derived necessary conditions for optimality and *analytically* studied a few special cases
• We now focus on *numerical* techniques to solve two-point boundary value problems; popular methods:
  • Indirect shooting method
  • Indirect collocation method
Let’s revisit our example...

Find optimal control $u(t)$ to steer the system

$$\ddot{x}(t) = u(t)$$

from $x(0) = 10, \dot{x}(0) = 0$ to the origin $x(t_f) = 0, \dot{x}(t_f) = 0$, and to minimize

$$J = \frac{1}{2} \alpha t_f^2 + \frac{1}{2} \int_{t_0}^{t_f} b u^2(t) dt, \quad \alpha, b > 0$$

• Solution: optimal time is

$$t_f = \left( \frac{1800b}{\alpha} \right)^{1/5}$$
Indirect methods: practical aspects

To obtain solution to the necessary conditions for optimality, one needs to solve two-point boundary value problems

• In python, we’ll be using `scipy.integrate.solve_bvp` to solve problems in “standard” form

\[
\dot{z} = g(z, t, p), \quad BC\left(z(t_0), z(t_f)\right) = 0
\]

where \(p\) are extra variables that can also be optimized

• Syntax: `sol = solve_bvp(fun, bc, t, z, p=None)`

Example: \(\dot{z}_1 = z_2, \quad \dot{z}_2 = -|z_1|, \quad z_1(0) = 0, \quad z_1(4) = -2\)

*`solve_bvp` uses a collocation formula (three-stage Lobatto)*
Extensions

• What about problems whose necessary conditions to not fit directly the “standard” form (e.g., free end time problems)?

• Handy tricks exist to convert problems into standard form:

**Important case:** free final time

1. Rescale time so that \( \tau = t/t_f \), then \( \tau \in [0,1] \)

2. Change derivatives \( \frac{d}{d\tau} := t_f \frac{d}{dt} \)

3. Introduce dummy state \( r \) that corresponds to \( t_f \) with dynamics \( \dot{r} = 0 \)

4. Replace all instances of \( t_f \) with \( r \)
Example

Find optimal control $u(t)$ to steer the system $\ddot{x}(t) = u(t)$ from $x(0) = 10, \dot{x}(0) = 0$ to the origin $x(t_f) = 0, \dot{x}(t_f) = 0$, and to minimize

$$J = \frac{1}{2} \alpha t_f^2 + \frac{1}{2} \int_{t_0}^{t_f} b u^2(t) dt, \quad \alpha, b > 0$$

Solution

1. Define state as $z = [x, p, r]$

2. BC are: $x_1(0) = 10, x_2(0) = 0, x_1(t_f) = 0, x_2(t_f) = 0, -\frac{p_2(t_f)^2}{2b} + \alpha t_f = 0$

3. BVP becomes:

$$\frac{dz}{dt} = t_f \frac{dz}{dt} = z_5 \begin{bmatrix} A & -B[0 \ 1]/b & 0 \\ 0 & -A' & 0 \\ 0 & 0 & 0 \end{bmatrix} z$$

4. BC become $z_1(0) = 10, z_2(0) = 0, z_1(1) = 0, z_2(1) = 0, -\frac{z_4(1)^2}{2b} + \alpha z_5(1) = 0$
Next time

• Direct methods