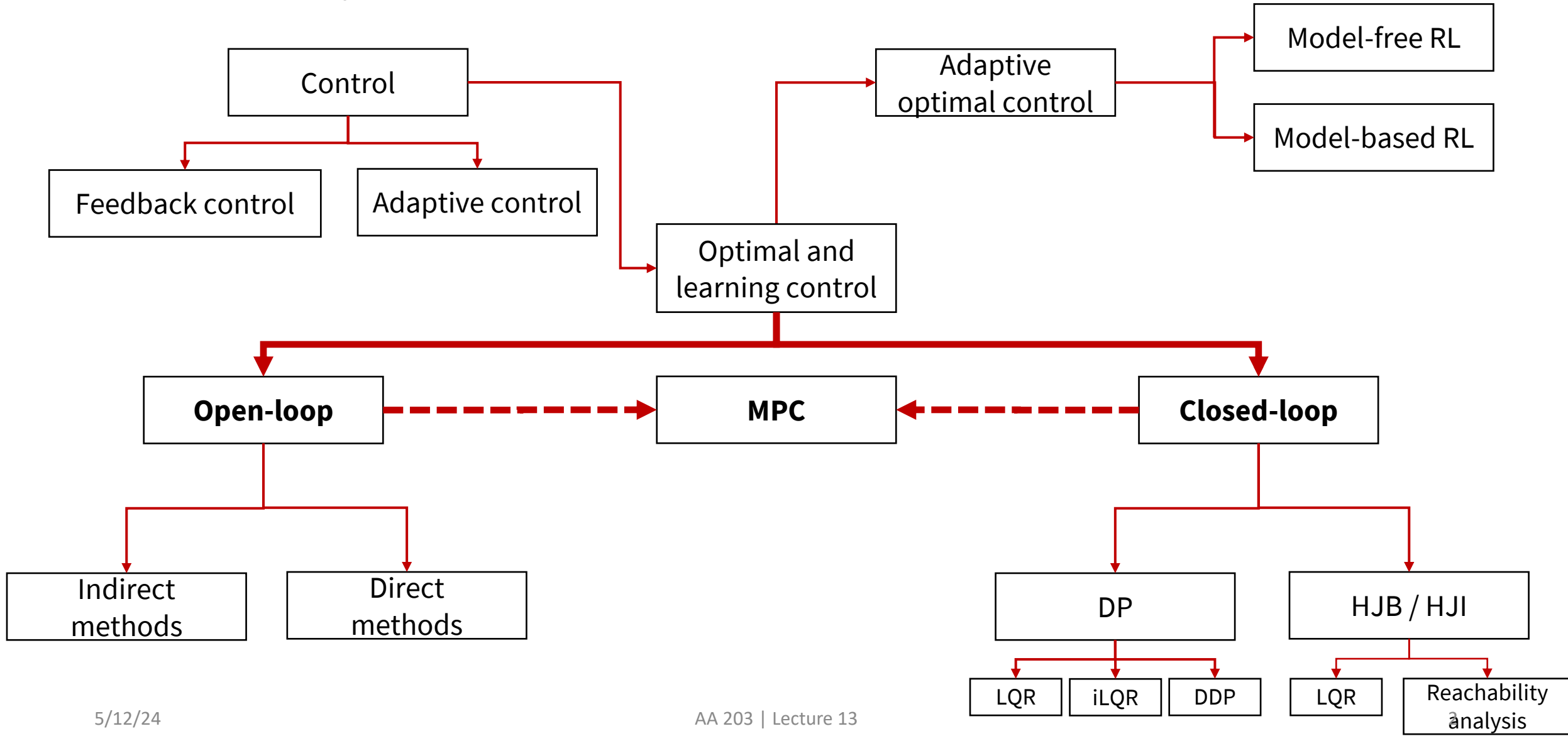


AA203

Optimal and Learning-based Control

Persistent feasibility of MPC (cont'd), stability of MPC

Roadmap



Model predictive control

- Persistent feasibility of MPC (cont'd)
- Stability of MPC
- Reading:
 - F. Borrelli, A. Bemporad, M. Morari. *Predictive Control for Linear and Hybrid Systems*, 2017.
 - J. B. Rawlings, D. Q. Mayne, M. M. Diehl. *Model Predictive Control: Theory, Computation, and Design*, 2017.

Persistent feasibility theorem

- Feasibility theorem: if set X_f is a *control invariant set* for system:

$$\mathbf{x}(t + 1) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(t) \in X, \quad \mathbf{u}(t) \in U, \quad t \geq 0$$

then the MPC law is persistently feasible

Persistent feasibility theorem

- Proof

1. Define “truncated” feasibility set at step $N - 1$:

$$X_{N-1} := \{\mathbf{x}_{N-1} \in X \mid \exists \mathbf{u}_{N-1} \text{ such that } \mathbf{x}_{N-1} \in X, \mathbf{u}_{N-1} \in U, \\ \mathbf{x}_N \in X_f \text{ where } \mathbf{x}_N = A\mathbf{x}_{N-1} + B\mathbf{u}_{N-1}\}$$

2. Due to the terminal constraint

$$A\mathbf{x}_{N-1} + B\mathbf{u}_{N-1} = \mathbf{x}_N \in X_f$$

Persistent feasibility theorem

- Proof

3. Since X_f is a control invariant set, there exists a $\mathbf{u} \in U$ such that

$$\mathbf{x}^+ = A\mathbf{x}_N + B\mathbf{u} \in X_f$$

4. The above is indeed the requirement to belong to set X_{N-1}
5. Thus, $A\mathbf{x}_{N-1} + B\mathbf{u}_{N-1} = \mathbf{x}_N \in X_{N-1}$
6. We have just proved that X_{N-1} is control invariant
7. Repeating this argument, one can recursively show that $X_{N-2}, X_{N-3}, \dots, X_1$ are control invariant, and the persistent feasibility lemma then applies

Practical aspects of persistent feasibility

- The terminal set X_f is introduced *artificially* for the sole purpose of leading to a *sufficient condition* for persistent feasibility
- We want it to be large so that it does not compromise closed-loop performance
- Though it is simplest to choose $X_f = \{0\}$, this is generally undesirable
- We'll discuss better choices later

Stability of MPC

- Persistent feasibility does not guarantee that the closed-loop trajectories converge towards the desired equilibrium point
- One of the most popular approaches to guarantee persistent feasibility and stability of the MPC law makes use of a control invariant terminal set X_f for feasibility, and of a terminal function $p(\cdot)$ for stability
- To prove stability, we leverage the tool of **Lyapunov stability theory**

Lyapunov stability theory

- **Lyapunov theorem:** Consider the equilibrium point $\mathbf{x} = 0$ for the autonomous system $\{\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)\}$ (with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$). Let $\Omega \subset \mathbb{R}^n$ be a closed, bounded, positively invariant set containing the origin. Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function, continuous at the origin, such that

$$V(\mathbf{0}) = 0 \text{ and } V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \Omega \setminus \{\mathbf{0}\}$$

$$V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k) < 0 \quad \forall \mathbf{x}_k \in \Omega \setminus \{\mathbf{0}\}$$

Then $\mathbf{x} = 0$ is asymptotically stable in Ω .

- The idea is to show that with appropriate choices of X_f and $p(\cdot)$, J_0^* is a Lyapunov function for the closed-loop system

MPC stability theorem

- **MPC stability theorem** (for quadratic cost): Assume

A0: $Q = Q^T \succ 0, R = R^T \succ 0, P \succ 0$

A1: Sets X, X_f , and U contain the origin in their interior and are closed

A2: $X_f \subseteq X$ is control invariant and bounded

A3:
$$\min_{\mathbf{u} \in U, A\mathbf{x} + B\mathbf{u} \in X_f} \left(-p(\mathbf{x}) + c(\mathbf{x}, \mathbf{u}) + p(A\mathbf{x} + B\mathbf{u}) \right) \leq 0, \forall \mathbf{x} \in X_f$$

Then, the origin of the closed-loop system is asymptotically stable with domain of attraction X_0 .

MPC stability theorem

- Proof:

1. Note that, by assumption A2, persistent feasibility is guaranteed for *any* P, Q, R
2. We want to show that J_0^* is a Lyapunov function for the closed-loop system $\mathbf{x}(t + 1) = \mathbf{f}_{cl}(\mathbf{x}(t))$, with respect to the equilibrium $\mathbf{f}_{cl}(\mathbf{0}) = \mathbf{0}$ (the origin is indeed an equilibrium as $\mathbf{0} \in X, \mathbf{0} \in U$, and the cost is positive for any non-zero control sequence)
3. X_0 is bounded and closed (follows from assumption on X_f)
4. $J_0^*(\mathbf{0}) = 0$ (value is nonnegative by construction, and 0 is achievable)

MPC stability theorem

- Proof:

5. $J_0^*(\mathbf{x}) > 0$ for all $\mathbf{x} \in X_0 \setminus \{\mathbf{0}\}$

6. Next we show the decay property. Since the setup is time-invariant, we can study the decay property between $t = 0$ and $t = 1$

- Let $\mathbf{x}(0) \in X_0$, let $U_0^{[0]} = [\mathbf{u}_0^{[0]}, \mathbf{u}_1^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}]$ be the optimal control sequence, and let $[\mathbf{x}(0), \mathbf{x}_1^{[0]}, \dots, \mathbf{x}_N^{[0]}]$ be the corresponding trajectory
- After applying $\mathbf{u}_0^{[0]}$, one obtains $\mathbf{x}(1) = A\mathbf{x}(0) + B\mathbf{u}_0^{[0]}$
- Consider the sequence of controls $[\mathbf{u}_1^{[0]}, \mathbf{u}_2^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}, \mathbf{v}]$, where $\mathbf{v} \in U$, and the corresponding state trajectory is $[\mathbf{x}(1), \mathbf{x}_2^{[0]}, \dots, \mathbf{x}_N^{[0]}, A\mathbf{x}_N^{[0]} + B\mathbf{v}]$

MPC stability theorem

- Since $\mathbf{x}_N^{[0]} \in X_f$ (by terminal constraint), and since X_f is control invariant,
 $\exists \bar{\mathbf{v}} \in U$ such that $A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}} \in X_f$
- With such a choice of $\bar{\mathbf{v}}$, the sequence $[\mathbf{u}_1^{[0]}, \mathbf{u}_2^{[0]}, \dots, \mathbf{u}_{N-1}^{[0]}, \bar{\mathbf{v}}]$ is feasible for the MPC optimization problem at time $t = 1$
- Since this sequence is not necessarily optimal

$$J_0^*(\mathbf{x}(1)) \leq p\left(A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}}\right) + \sum_{k=1}^{N-1} c\left(\mathbf{x}_k^{[0]}, \mathbf{u}_k^{[0]}\right) + c\left(\mathbf{x}_N^{[0]}, \bar{\mathbf{v}}\right)$$

MPC stability theorem

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$$J_0^*(\mathbf{x}(1)) \leq p(A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}}) + \sum_{k=1}^{N-1} c(\mathbf{x}_k^{[0]}, \mathbf{u}_k^{[0]}) + c(\mathbf{x}_N^{[0]}, \bar{\mathbf{v}}) \\ + p(\mathbf{x}_N^{[0]}) - p(\mathbf{x}_N^{[0]}) + c(\mathbf{x}(0), \mathbf{u}_0^{[0]}) - c(\mathbf{x}(0), \mathbf{u}_0^{[0]})$$

MPC stability theorem

- Equivalently

$$J_0^*(\mathbf{x}(1)) \leq p \left(A\mathbf{x}_N^{[0]} + B\bar{\mathbf{v}} \right) + J_0^*(\mathbf{x}(0)) - p \left(\mathbf{x}_N^{[0]} \right) - c \left(\mathbf{x}(0), \mathbf{u}_0^{[0]} \right) + c(\mathbf{x}_N^{[0]}, \bar{\mathbf{v}})$$

- Since $\mathbf{x}_N^{[0]} \in X_f$, by assumption A3, we can select $\bar{\mathbf{v}}$ such that

$$J_0^*(\mathbf{x}(1)) \leq J_0^*(\mathbf{x}(0)) - c \left(\mathbf{x}(0), \mathbf{u}_0^{[0]} \right)$$

- Since $c \left(\mathbf{x}(0), \mathbf{u}_0^{[0]} \right) > 0$ for all $\mathbf{x}(0) \in X_0 \setminus \{0\}$,

$$J_0^*(\mathbf{x}(1)) - J_0^*(\mathbf{x}(0)) < 0$$

- The last step is to prove continuity; details are omitted and can be found in Borrelli, Bemporad, Morari, 2017
- Note: A2 is used to guarantee persistent feasibility; this assumption can be replaced with an assumption on the horizon N

How to choose X_f and P ?

- Case 1: assume A is asymptotically stable
 - Set X_f as the maximally positive invariant set O_∞ for system $\mathbf{x}(t + 1) = A\mathbf{x}(t)$, $\mathbf{x}(t) \in X$
 - X_f is a control invariant set for system $\mathbf{x}(t + 1) = A\mathbf{x}(t) + B\mathbf{u}(t)$, as $\mathbf{u} = 0$ is a feasible control
 - As for stability, $\mathbf{u} = 0$ is feasible and $A\mathbf{x} \in X_f$ if $\mathbf{x} \in X_f$, thus assumption A3 becomes

$$-\mathbf{x}^T P \mathbf{x} + \mathbf{x}^T Q \mathbf{x} + \mathbf{x}^T A^T P A \mathbf{x} \leq 0, \text{ for all } \mathbf{x} \in X_f,$$

which is true since, due to the fact that A is asymptotically stable,

$$\exists P > 0 \mid -P + Q + A^T P A = 0 \quad (\text{Lyapunov Equation})$$

 Cost-to-go/value function

How to choose X_f and P ?

- Case 2: general case (e.g., if A is open-loop unstable)
 - Let F_∞ be the optimal gain for the infinite-horizon LQR controller
 - Set X_f as the maximal positive invariant set for system

$$\mathbf{x}(t + 1) = (A + BF_\infty)\mathbf{x}(t)$$

(with constraints $\mathbf{x}(t) \in X$, and $F_\infty\mathbf{x}(t) \in U$)

- Set P as the solution P_∞ to the discrete-time Riccati equation, i.e., the value function via LQR

$$-P + Q + A^T P A - (A^T P B)(R + B^T P B)^{-1}(B^T P A) = 0$$

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- **Note: both cases as presented are just (suboptimal) choices!**

Next time

- Explicit MPC
- Practical aspects of MPC
- Robust MPC