

AA 203

Optimal and Learning-Based Control

Pontryagin's maximum principle and indirect methods

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Agenda

1. Geometry and generalizations of first-order NOCs
2. Weak Pontryagin maximum principle in discrete-time
3. Weak Pontryagin maximum principle in continuous-time
4. Pontryagin maximum principle in continuous-time
5. Indirect methods for optimal control
6. Time-optimal control problems

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Review: First-order NOCs

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && f(x) \\ & \text{subject to} && h(x) = 0 \\ & && g(x) = 0 \end{aligned} \quad L(x, \lambda, \mu) := f(x) + \lambda^\top h(x) + \mu^\top g(x)$$

Theorem (First-order NOCs)

Suppose $x^* \in \mathbb{R}^n$ is a local minimum of $f \in C^1(\mathbb{R}^n, \mathbb{R})$ subject to $h(x) = 0$ and $g(x) = 0$ with $h \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ and $g \in C^1(\mathbb{R}^n, \mathbb{R}^r)$. Moreover, assume

$$\{ \nabla h_i(x^*) \}_{i=1}^m \cup \{ \nabla g_j(x^*) \}_{j \in A_g(x^*)}$$

are linearly independent. Then there exist unique $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^r$ such that

$$\nabla_x L(x^*, \lambda, \mu) = 0, \quad \mu = 0, \quad \mu_j = 0, \quad j \notin A_g(x^*),$$

The assumption on the constraint gradients is known as the *linear independence constraint qualification (LICQ)*.

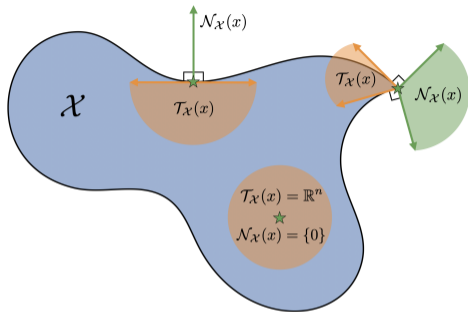
Geometry of first-order NOCs

Tangent cone $T_X(x)$ “vectors that stay in X ”

Normal cone $N_X(x)$ “vectors that leave X ”

If x is a local minimum of f over X , then $-\nabla f(x) \in N_X(x)$, i.e., there is no feasible component of $-\nabla f(x)$ that would allow us to locally decrease $f(x)$.

For convenience, we write “ $-\nabla f(x) \in N_X(x)$ ”. In other literature, you may see “ $-\nabla f(x) \in T_X(x)$ ”.



If $X = \{x \in \mathbb{R}^n \mid h(x) = 0, g(x) \leq 0\}$ and the LICQ holds at $x \in X$, then

$$T_X(x) = \left\{ d \in \mathbb{R}^n \mid \frac{\partial h}{\partial x}(x) d = 0, g_j(x)^\top d \leq 0, j \in A_g(x) \right\}$$

$$N_X(x) = \left\{ v \in \mathbb{R}^n \mid v = \frac{\partial h}{\partial x}(x)^\top \lambda + \frac{\partial g}{\partial x}(x)^\top \mu, \mu \geq 0, \mu_j = 0, j \notin A_g(x) \right\}$$

Example: A problem with linearly dependent constraints

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad f(x) := x_1 + x_2$$

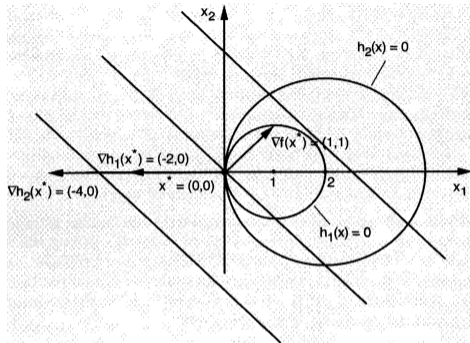
$$\text{subject to } h_1(x) := (x_1 - 1)^2 + x_2^2 - 1 = 0$$

$$h_2(x) := (x_1 - 2)^2 + x_2^2 - 4 = 0$$

At the only feasible point $x^* = 0$, we have

$$f(x^*) = (1, 1)$$

$$h_1(x^*) = (-2, 0), \quad h_2(x^*) = (-4, 0)$$



The constraint gradients are linearly dependent (i.e., the LICQ does not hold), so we cannot write $\nabla f(x^*) + \lambda_1 \nabla h_1(x^*) + \lambda_2 \nabla h_2(x^*) = 0$.

In essence, the constraints “pinch together” so that just one x is feasible, regardless of the objective value.

Theorem (Fritz John first-order NOCs)

Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$, $h \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, and $g \in C^1(\mathbb{R}^n, \mathbb{R}^r)$. Suppose $x^* \in \mathbb{R}^n$ is a local minimum of the problem

$$\begin{aligned} & \underset{x \in S}{\text{minimize}} && f(x) \\ & \text{subject to} && h(x) = 0 \\ & && g(x) \leq 0 \end{aligned}$$

Then there exist $(\eta, \lambda, \mu) \in \{0, 1\} \times \mathbb{R}^m \times \mathbb{R}^r$ such that

$$\begin{aligned} & \eta = 0 && \text{non-triviality} \\ & - \nabla_x L_\eta(x^*, \lambda, \mu) = 0 && \text{stationarity} \\ & \mu_j \geq 0, \mu_j g_j(x^*) = 0, \quad j \in \{1, 2, \dots, r\} && \text{complementarity} \end{aligned}$$

where $L_\eta(x, \lambda, \mu)$ is the partial Lagrangian

$$L_\eta(x, \lambda, \mu) := \eta f(x) + \lambda^\top h(x) + \mu^\top g(x).$$

Fritz John first-order NOCs

Theorem (Fritz John first-order NOCs)

If x^* is a local minimum, there exist $(\eta, \lambda, \mu) \in \{0, 1\} \times \mathbb{R}^m \times \mathbb{R}^r$ such that

$$\begin{aligned} & \eta = 0 \\ & - \nabla_x L_\eta(x^*, \lambda, \mu) = 0 \\ & \mu_j = 0, \mu_j g_j(x^*) = 0, \quad j \in \{1, 2, \dots, r\} \end{aligned}$$

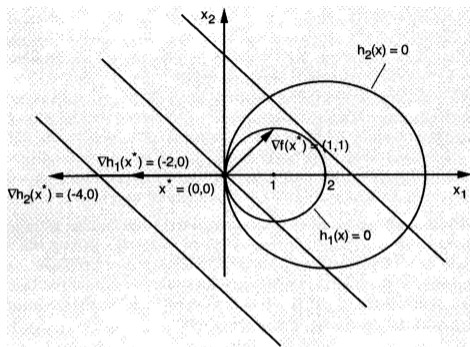
where $L_\eta(x, \lambda, \mu)$ is the partial Lagrangian

$$L_\eta(x, \lambda, \mu) := \eta f(x) + \lambda^T h(x) + \mu^T g(x).$$

The “abnormal case” $\eta = 0$ yields necessary conditions independent of the objective f .

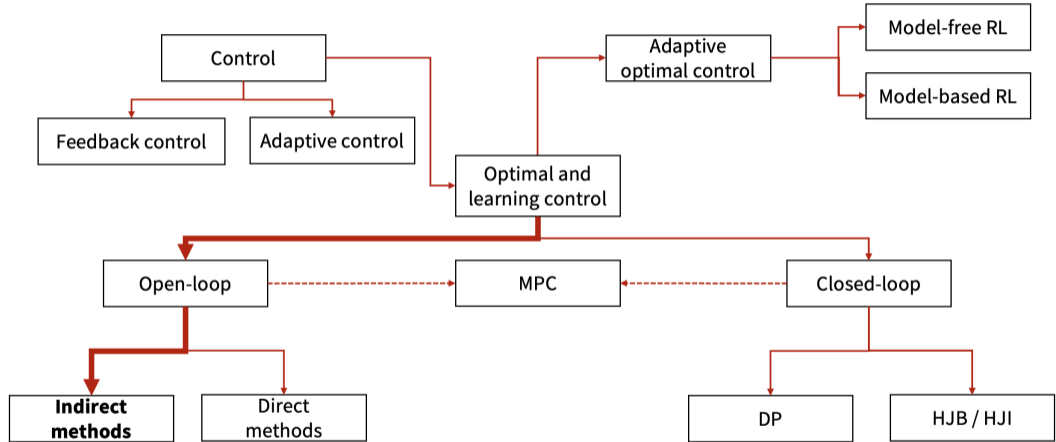
Corollary

If $S = \mathbb{R}^n$ and the LICQ holds, then $\eta = 1$ and $-\nabla_x L_1(x^*, \lambda, \mu) = 0$.



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Course overview



Optimal control problem (discrete-time)

Consider the discrete-time optimal control problem (OCP)

$$\begin{aligned} & \underset{x,u}{\text{minimize}} && \ell_T(x_T) + \sum_{t=0}^{T-1} \ell(t, x_t, u_t) && \text{cost (terminal + stage)} \\ & \text{subject to} && x_{t+1} = f(t, x_t, u_t), \quad t \in \{0, 1, \dots, T-1\} && \text{dynamical feasibility} \\ & && x_0 = \bar{x}_0 && \text{initial condition} \\ & && x_T \in X_T && \text{terminal condition} \\ & && u_t \in U, \quad t \in \{0, 1, \dots, T-1\} && \text{input constraints} \end{aligned}$$

An optimal control $u = \{u_t\}_{t=0}^{T-1}$ for a specific initial state \bar{x}_0 is an *open-loop* input.

An optimal control of the form $u_t = \pi(t, x_t)$ is a *closed-loop* input.

Lagrangian, Hamiltonian, and the adjoint equation (discrete-time)

The partial Lagrangian is

$$\begin{aligned} L_\eta(x, u, p) &= \eta \ell_T(x_T) + \underbrace{p_0^\top (x_0 - \bar{x}_0)}_{\text{initial condition}} + \sum_{t=0}^{T-1} \left(\eta \ell(t, x_t, u_t) + \underbrace{p_{t+1}^\top (x_{t+1} - f(t, x_t, u_t))}_{\text{dynamical feasibility}} \right), \\ &= \eta \ell_T(x_T) + p_0^\top (x_0 - \bar{x}_0) + \sum_{t=0}^{T-1} (p_{t+1}^\top x_{t+1} - H_\eta(t, x_t, u_t, p_{t+1})) \end{aligned}$$

with normality $\eta \in \{0, 1\}$, Lagrange multipliers $\{p_t\}_{t=0}^T \in \mathbb{R}^n$, and *Hamiltonian*

$$H_\eta(t, x, u, p) := p^\top f(t, x, u) - \eta \ell(t, x, u).$$

Setting $\frac{\partial L}{\partial x_t}(x_t, u_t) = 0$ for $t \in \{0, 1, \dots, T-1\}$ yields

$$p_t = - \frac{\partial H_\eta(t, x_t, u_t, p_{t+1})}{\partial x_t}, \quad t \in \{0, 1, \dots, T-1\},$$

which is a *backwards recursion* for the *adjoint* or *co-state* p_t .

Transversality and the maximum condition (discrete-time)

The partial Lagrangian is

$$L_\eta(x, u, p) = \eta \ell_T(x_T) + p_0^\top (x_0 - \bar{x}_0) + \sum_{t=0}^{T-1} (p_{t+1}^\top x_{t+1} - H_\eta(t, x_t, u_t, p_{t+1}))$$

where we left out $x_T \in X_T$ and $u_t \in U$. Setting $-x_T L_\eta(x, u) \in x_T X_T$ yields the *transversality condition*

$$-p_T - \eta \ell_T(x_T) \in x_T X_T,$$

and setting $-u_t L(x, u) \in u_t U$ yields the *weak maximum condition*

$$u H_\eta(t, x_t, u_t, p_{t+1}) \in u_t U, \quad t \in \{0, 1, \dots, T-1\}.$$

We refer to this condition as “weak” since it is a necessary, but not sufficient condition for a solution of the problem

$$\underset{u \in U}{\text{maximize}} H_\eta(t, x_t, u, p_{t+1}).$$

Pontryagin maximum principle (discrete-time)

Collect these necessary conditions together to get the Pontryagin maximum principle (PMP).

Theorem (Pontryagin maximum principle (discrete-time))

Let (x, u) be a local minimum of the discrete-time OCP with terminal set X_T and control set U . Then $\eta \in \{0, 1\}$ and $\{p_t\}_{t=0}^T \in \mathbb{R}^n$ exist such that

$$(\eta, p_0, p_1, \dots, p_T) = 0 \quad \text{non-triviality}$$

$$p_t = -\frac{\partial H_\eta}{\partial x}(t, x_t, u_t, p_{t+1}), \quad t \in \{0, 1, \dots, T-1\} \quad \text{adjoint equation}$$

$$-p_T - \eta \frac{\partial \ell_T}{\partial x}(x_T) \in \frac{\partial X_T}{\partial x_T} \quad \text{transversality}$$

$$\frac{\partial H_\eta}{\partial u}(t, x_t, u_t, p_{t+1}) \in \frac{\partial U}{\partial u_t}, \quad t \in \{0, 1, \dots, T-1\} \quad \text{maximum condition (weak)}$$

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Optimal control problem (continuous-time)

Consider the continuous-time optimal control problem (OCP)

$$\begin{array}{ll} \underset{x,u}{\text{minimize}} & \ell_T(x(T)) + \int_0^T \ell(t, x(t), u(t)) dt & \text{cost (terminal + stage)} \\ \text{subject to} & \dot{x}(t) = f(t, x(t), u(t)), \quad t \in [0, T] & \text{dynamical feasibility} \\ & x(0) = x_0 & \text{initial condition} \\ & x(T) \in X_T & \text{terminal condition} \\ & u(t) \in U, \quad t \in [0, T] & \text{input constraints} \end{array}$$

An optimal control $u^*(t)$ for a specific initial state x_0 is an *open-loop* input.

An optimal control of the form $u^*(t) = \pi^*(t, x(t))$ is a *closed-loop* input.

Consider piecewise continuous trajectories such that $x(t) = x(t_k)$ and $u(t) = u(t_k)$ for $t \in [t_k, t_{k+1})$, with $k \in \{0, 1, \dots, N-1\}$, $t_0 = 0$ and $t_N = T$.

Define $\Delta t_k := t_{k+1} - t_k$ such that $\Delta t_k > 0$ for all $k \in \{0, 1, \dots, N-1\}$.

Consider the discretized OCP

$$\begin{aligned} & \underset{x, u}{\text{minimize}} \quad \ell_T(x(t_N)) + \sum_{k=0}^{N-1} \Delta t_k \ell(t_k, x(t_k), u(t_k)) \\ & \text{subject to} \quad x(t_{k+1}) = x(t_k) + \Delta t_k f(t_k, x(t_k), u(t_k)), \quad k \in \{0, 1, \dots, N-1\} \\ & \quad \quad \quad x(t_0) = x_0 \\ & \quad \quad \quad x(t_N) \in X_T \\ & \quad \quad \quad u(t_k) \in U, \quad k \in \{0, 1, \dots, N-1\} \end{aligned}$$

Use the discrete-time PMP on a local minimum (x^*, u^*) of the discretized OCP to get

$$\begin{aligned}
 & (\eta, p(t_0), p(t_1), \dots, p(t_N)) = 0 \\
 & - \frac{(p(t_{k+1}) - p(t_k))}{\Delta t_k} = -x^T H_\eta(t_k, x(t_k), u(t_k), p(t_{k+1})), \quad k \in \{0, 1, \dots, N-1\} \\
 & - p(t_N) - \eta \ell_T(x(t_N)) = x^T(t_N) X_T \\
 & u^T H_\eta(t_k, x(t_k), u(t_k), p(t_{k+1})) = u^T U, \quad k \in \{0, 1, \dots, N-1\}
 \end{aligned}$$

where we use the *continuous-time Hamiltonian*

$$H_\eta(t, x, u, p) := p^T f(t, x, u) - \eta \ell(t, x, u).$$

Pontryagin maximum principle (continuous-time, weak)

The above conditions suggest the following continuous-time PMP as $\Delta t_k \rightarrow 0$.

Theorem (Pontryagin maximum principle (continuous-time, weak))

Let (x^*, u^*) be a local minimum of the continuous-time optimal control problem with terminal set X_T and control set U . Then $\eta \in \{0, 1\}$ and $p : [0, T] \rightarrow \mathbb{R}^n$ exist such that

$$(\eta, p(t)) \neq 0 \quad \text{non-triviality}$$

$$-\dot{p}(t) = -\frac{\partial H}{\partial x}(t, x^*(t), u^*(t), p(t)), \quad t \in [0, T] \quad \text{adjoint equation}$$

$$-p(T) - \eta \frac{\partial \ell_T}{\partial x}(x^*(T)) \in \frac{\partial \ell_T}{\partial x}(x^*(T)) \quad \text{transversality}$$

$$H(t, x^*(t), u^*(t), p(t)) \leq H(t, x^*(t), u(t), p(t)) \quad \forall u(t) \in U, \quad t \in [0, T] \quad \text{maximum condition (weak)}$$

" $(\eta, p(t)) \neq 0$ " means there exists at least one $t \in [0, T]$ such that $(\eta, p(t)) \neq 0$.

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Recall that (x^*, u^*) is a *local minimum* of $J(x, u)$ if there exists $\varepsilon > 0$ such that $J(x^*, u^*) \leq J(x, u)$ for all (x, u) in the ε -sized norm ball around (x^*, u^*) .

In using the discrete-time PMP as a heuristic to obtain the continuous-time PMP, we are implicitly using the C^0 -norm for both x and u , i.e.,

$$\|x - x^*\|_{C^0} := \max_{t \in [0, T]} |x(t) - x^*(t)|, \quad \|u - u^*\|_{C^0} := \max_{t \in [0, T]} |u(t) - u^*(t)|.$$

We can strengthen the continuous-time PMP if we use the C^0 -norm for x and the L^1 -norm for u , i.e.,

$$\|x - x^*\|_{C^0} := \max_{t \in [0, T]} |x(t) - x^*(t)|, \quad \|u - u^*\|_{L^1} := \int_0^T |u(t) - u^*(t)| dt.$$

Strengthening the maximum condition via needle perturbations

In general, the L^1 -norm ball for u allows for *large pointwise variations* at each time t . Suppose the control set U is bounded, i.e., $u - v \leq c$ for all $u, v \in U$ and some $c > 0$.

Given some $u : [0, T] \rightarrow U$, any $\tau \in [0, T)$ and $\varepsilon > 0$ such that $[\tau, \tau + \varepsilon] \subset [0, T]$, and any $v \in U$, define

$$u(t) = \begin{cases} v, & t \in [\tau, \tau + \varepsilon) \\ u(t), & t \in [0, \tau) \cup [\tau + \varepsilon, T] \end{cases}$$

This is a *spatial needle perturbation* of $u(t)$. Then it can be shown that

$$\|u - u\|_{L^1} := \int_0^T |u(t) - u(t)| dt = \int_{\tau}^{\tau + \varepsilon} |v - u(t)| dt \leq \int_{\tau}^{\tau + \varepsilon} c dt = \varepsilon c.$$
$$x(T) = x(T) + \varepsilon d, \quad d \in T_{X_T}(x(T))$$

for small enough ε . Overall, a large spatial perturbation in $u(t)$ can correspond to small feasible perturbations to both x and u .

Pontryagin maximum principle (continuous-time)

The possibility of large spatial control perturbations still corresponding to “feasible neighbours” of (x, u) suggests the following strengthened PMP.

Theorem (Pontryagin maximum principle (continuous-time))

Let (x, u) be a local minimum (using the C^0 -norm and L^1 -norm, respectively) of the continuous-time OCP with terminal set X_T and bounded control set U . Then $\eta \in \{0, 1\}$ and $p : [0, T] \rightarrow \mathbb{R}^n$ exist such that

$$(\eta, p(t)) \neq 0 \quad \text{non-triviality}$$

$$-\dot{p}(t) = -x H_\eta(t, x(t), u(t), p(t)), \quad t \in [0, T] \quad \text{adjoint equation}$$

$$-p(T) - \eta \ell_T(x(T)) \in x(T)^\perp \cap X_T^\perp \quad \text{transversality}$$

$$H_\eta(t, x(t), u(t), p(t)) = \sup_{u \in U} H_\eta(t, x(t), u, p(t)), \quad t \in [0, T] \quad \text{maximum condition}$$

A rigorous proof relies on variational calculus ([Liberzon, 2012](#); [Clarke, 2013](#)).

Example: Minimum fuel for a control-affine system

Consider the continuous-time OCP

$$\underset{x,u}{\text{minimize}} \int_0^T \sum_{j=1}^m \alpha_j / u_j(t) dt$$

$$\text{subject to } \dot{x}(t) = a(t, x(t)) + \sum_{j=1}^m u_j(t) b_j(t, x(t)), \quad t \in [0, T]$$

$$x(0) = x_0$$

$$x(T) = 0$$

$$- \bar{u} \leq u(t) \leq \bar{u}, \quad t \in [0, T]$$

where $\bar{u} > 0$. The Hamiltonian is

$$H_\eta(t, x, u, p) = p^\top \left(a(t, x) + \sum_{j=1}^m u_j b_j(t, x) \right) - \eta \sum_{j=1}^m \alpha_j / u_j$$

Example: Minimum fuel for a control-affine system

The Hamiltonian is

$$H_\eta(t, x, u, p) = a(t, x)^\top p + \sum_{j=1}^m (u_j b_j(t, x)^\top p - \eta \alpha_j |u_j|)$$

The adjoint equation is

$$\dot{p} = - \frac{\partial}{\partial x} H_\eta(t, x, u, p) = - \frac{\partial a}{\partial x}(t, x) p - \sum_{j=1}^m u_j \frac{\partial b_j}{\partial x}(t, x) p$$

The maximum condition is

$$u_j = \arg \max_{u_j \in [-\bar{u}_j, \bar{u}_j]} (u_j b_j(t, x)^\top p - \eta \alpha_j |u_j|) = \begin{cases} -\bar{u}_j, & b_j(t, x)^\top p < -\eta \alpha_j \\ 0, & b_j(t, x)^\top p \in [-\eta \alpha_j, \eta \alpha_j] \\ \bar{u}_j, & b_j(t, x)^\top p > \eta \alpha_j \end{cases}$$

which for $\eta = 1$ is an example of “bang-off-bang” control.

Example: Minimum fuel for a control-affine system

Assume $\eta = 1$, i.e., the “normal” case. Altogether, we have the *boundary value problem (BVP)*

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} a(t, x) + \sum_{j=1}^m u_j b_j(t, x) \\ -\frac{\partial a}{\partial x}(t, x)p - \sum_{j=1}^m u_j \frac{\partial b_j}{\partial x}(t, x)p \end{pmatrix}, \quad u_j = \begin{cases} -\bar{u}_j, & b_j(t, x)^\top p < -\alpha_j \\ 0, & b_j(t, x)^\top p \in [-\alpha_j, \alpha_j] \\ \bar{u}_j, & b_j(t, x)^\top p > \alpha_j \end{cases},$$

with boundary conditions $x(0) = x_0$ and $x(T) = 0$.

Transversality did not factor into this problem, since the normal cone of the singleton $X_T = \{0\}$ is just \mathbb{R}^n (i.e., any direction “leaves” the terminal set).

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An *indirect method* generally focuses on solving the BVP

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} f(t, x, u) \\ -H_x(t, x, u, p) \end{pmatrix}, \quad x(0) = x_0, \quad h(x(T), p(T)) = 0.$$

where $h(x(T), p(T)) \in \mathbb{R}^n$. The *open-loop* optimal control candidate $u(t, x(t), p(t))$ is then extracted.

The boundary condition $h(x(T), p(T)) = 0$ is determined by the terminal set constraint $x(T) \in X_T$ and the transversality condition $-p(T) \in \eta \ell_T(x(T)) \times X_T$.

We are implicitly assuming an optimal control exists. Even then, there may be multiple local optima.

To solve the BVP

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} f(t, x, u) \\ -x H_{\eta}(t, x, u(t, x, p), p) \end{pmatrix}, \quad x(0) = x_0, \quad h(x(T), p(T)) = 0,$$

we consider the associated initial value problem (IVP)

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} f(t, x, u) \\ -x H_{\eta}(t, x, u(t, x, p), p) \end{pmatrix}, \quad x(0) = x_0, \quad p(0) = p_0.$$

We can integrate the IVP forward in time to get $x(T; p_0)$ and $p(T; p_0)$, which are parameterized by p_0 .

We can use a root-finding method (e.g., bisection search, Newton-Raphson method) to find p_0 such that $h(x(T; p_0), p(T; p_0)) = 0$. This is called *single shooting* and gives us a solution of the BVP.

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Consider the continuous-time OCP

$$\begin{aligned} & \underset{x, u, T}{\text{minimize}} && \ell_T(T, x(T)) + \int_0^T \ell(t, x(t), u(t)) dt && \text{cost (terminal + stage)} \\ & \text{subject to} && \dot{x}(t) = f(t, x(t), u(t)), \quad t \in [0, T] && \text{dynamical feasibility} \\ & && x(0) = x_0 && \text{initial condition} \\ & && x(T) \in X_T && \text{terminal condition} \\ & && u(t) \in U, \quad t \in [0, T] && \text{input constraints} \end{aligned}$$

The final time T is now a *free variable* (subject to $T \geq 0$).

Use the change of variables $t(s) = Ts$ with $s \in [0, 1]$ to get

$$\begin{aligned} & \underset{(x,t),(u,T)}{\text{minimize}} && \ell_T(t(1), x(1)) + T \int_0^1 \ell(t(s), x(s), u(s)) ds && \text{cost (terminal + stage)} \\ & \text{subject to} && \dot{x}(s) = Tf(t(s), x(s), u(s)), \quad \dot{t}(s) = T, \quad s \in [0, 1] && \text{dynamical feasibility} \\ & && x(0) = x_0, \quad t(0) = 0 && \text{initial condition} \\ & && x(1) \in X_T && \text{terminal condition} \\ & && u(s) \in U, \quad T \in [0, \infty), \quad s \in [0, 1] && \text{input constraints} \end{aligned}$$

To derive a new form of the PMP for time-optimal problems, we apply the fixed final time PMP to the problem above, where we treat t and T as a new state and input, respectively.

Deriving the time-optimal PMP

Applying the fixed final time PMP gives us the Hamiltonian

$$\tilde{H}_\eta(s, x, t, u, T, p, \lambda) = T(H(t, x, u, p) + \lambda),$$

where $H(t, x, u, p)$ is the usual Hamiltonian, and λ is the adjoint for the new "state" $t(s) = Ts$. Taking derivatives with respect to (x, t) yields the adjoint equations

$$\frac{dp}{ds} = -T \frac{\partial H}{\partial x}(t, x, u, p), \quad \frac{d\lambda}{ds} = -T \frac{\partial H}{\partial t}(t, x, u, p),$$

which by the chain rule with $\frac{dt}{ds} = T$ become

$$\dot{p} = - \frac{\partial H}{\partial x}(t, x, u, p), \quad \dot{\lambda} = - \frac{\partial H}{\partial t}(t, x, u, p).$$

Since t has no terminal constraint, we have the transversality conditions

$$-p(1) - \eta \frac{\partial \ell_T}{\partial x}(t(1), x(1)) = 0, \quad -\lambda(1) - \eta \frac{\partial \ell_T}{\partial t}(t(1), x(1)) = 0.$$

which after using $t = sT$ gives us

$$-p(T) - \eta \frac{\partial \ell_T}{\partial x}(T, x(T)) = 0, \quad -\lambda(T) - \eta \frac{\partial \ell_T}{\partial t}(T, x(T)) = 0.$$

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We are considering the absolute value norm for T , and $[0, \infty)$ is unbounded. So we use the maximum condition for u and the weak maximum condition for T to get

$$T \tilde{H}_\eta(t, x, u, p) \Big|_{T=0} = H(t, x, u, p) + \lambda = 0,$$

where we have assumed $T > 0$ to get that the normal cone is just $\{0\}$. Evaluating this condition at $t = T$ gives us

$$H(T, x(T), u(T), p(T)) = -\lambda(T) = \eta - t \ell_T(T, x(T)),$$

which is the additional boundary condition we need for free final time T .

Collecting all of the conditions we derived above gives us the free final time PMP.

Theorem (Pontryagin maximum principle (continuous-time, free final time))

Let (x^*, u^*, T^*) be a local minimum (using the C^0 -norm, L^1 -norm, and absolute value, respectively) of the continuous-time OCP with terminal set X_T , bounded control set U , and free final time $T \geq 0$. Then $\eta \in \{0, 1\}$ and $p : [0, T^*] \rightarrow \mathbb{R}^n$ exist such that

$$(\eta, p(t)) \neq 0 \quad \text{non-triviality}$$

$$-\dot{p}(t) = -\frac{\partial H_\eta}{\partial x}(t, x(t), u(t), p(t)), \quad t \in [0, T^*] \quad \text{adjoint equation}$$

$$-p(T^*) - \eta \frac{\partial \ell_T}{\partial x}(T^*, x(T^*)) \in \frac{\partial \ell_T}{\partial x}(T^*, x(T^*)) \quad \text{transversality}$$

$$H_\eta(t, x(t), u(t), p(t)) = \sup_{u \in U} H_\eta(t, x(t), u, p(t)), \quad t \in [0, T^*] \quad \text{maximum condition}$$

$$H_\eta(T^*, x(T^*), u(T^*), p(T^*)) = \eta \frac{\partial \ell_T}{\partial T}(T^*, x(T^*)) \quad \text{maximum condition (boundary)}$$

Direct methods for optimal control
(i.e., solving discretized optimal control problems directly)

- F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control*. Springer, 2013.
- D. Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton University Press, 2012.