AA203
Optimal and Learning-based Control

Stochastic DP, value iteration, policy iteration, stochastic LQR

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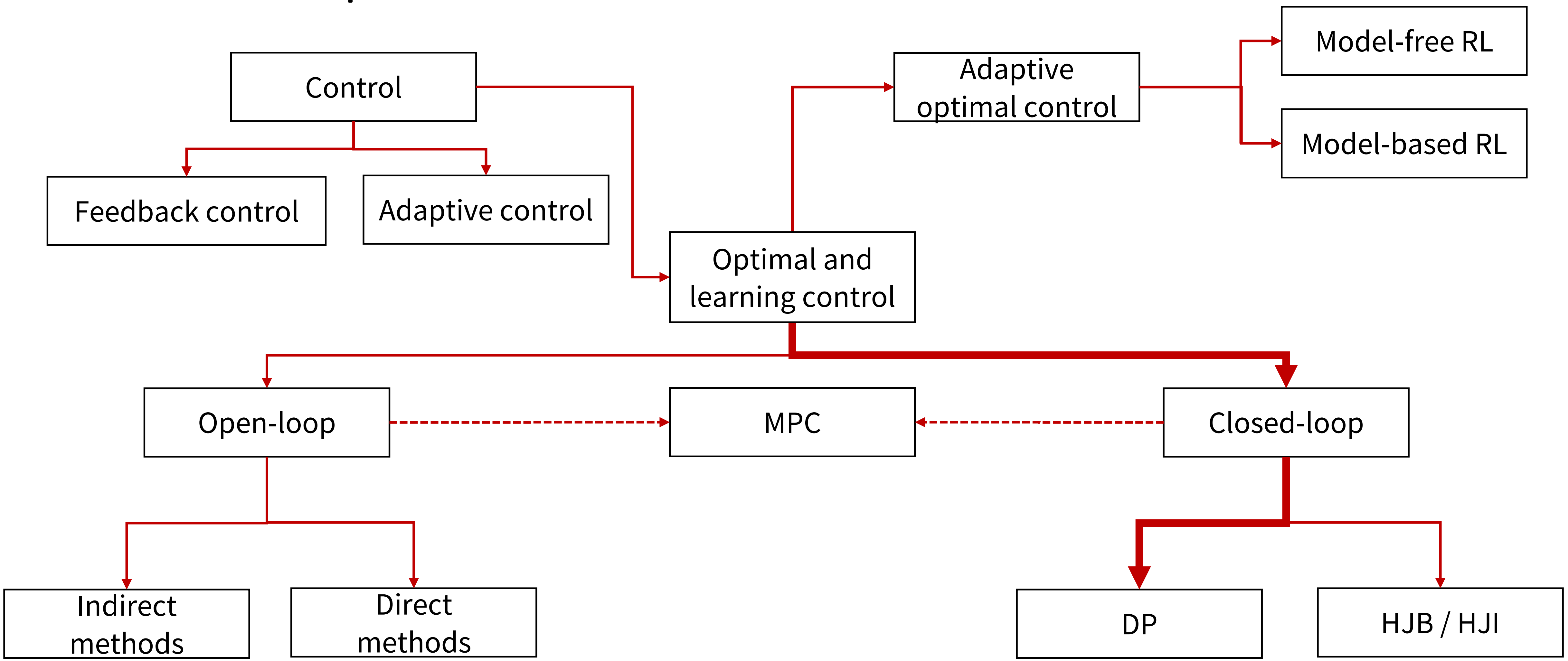
Roadmap



Control
Feedback control
Adaptive control
Optimal and learning control
Adaptive optimal control
Model-free RL
Model-based RL
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI

Stochastic optimal control problem (MDPs)

- System: $\boldsymbol{x}_{k+1} = f_k(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k), k = 0, \ldots, N-1$
- Control constraints: $\boldsymbol{u}_k \in U(\boldsymbol{x}_k)$
- Probability distribution: $P_k(\cdot \mid \boldsymbol{x}_k, \boldsymbol{u}_k)$ of $\boldsymbol{w}_k$
- Policies: $\pi = \{\pi_0 \ldots, \pi_{N-1}\}$, where $\boldsymbol{u}_k = \pi_k(\boldsymbol{x}_k)$
- Expected Cost:

$$J_\pi(\boldsymbol{x}_0) = E_{\boldsymbol{w}_k, k=0,\ldots,N-1}\left[g_N(\boldsymbol{x}_N) + \sum_{k=0}^{N-1} g_k(\boldsymbol{x}_k, \pi_k(\boldsymbol{x}_k), \boldsymbol{w}_k)\right]$$

- Stochastic optimal control problem

$$J^*(x_0) = \min_\pi J_\pi(\boldsymbol{x}_0)$$

Key points

- Discrete-time model
- Markovian model
- Objective: find optimal closed-loop policy
- Additive cost (central assumption)
- Risk-neutral formulation

Other communities use different notation: Powell, W. B. *AI, OR and control theory: A Rosetta Stone for stochastic optimization*. Princeton University, 2012.

Principle of optimality

- Let $\pi^* = \{\pi_0^*, \pi_1^*, \dots, \pi_{N-1}^*\}$ be an optimal policy
- Consider tail subproblem

$$E\left[g_N(\boldsymbol{x}_N) + \sum_{k=i}^{N-1} g_k(\boldsymbol{x}_k, \pi_k(\boldsymbol{x}_k), \boldsymbol{w}_k)\right]$$

and the tail policy $\{\pi_i^*, \dots, \pi_{N-1}^*\}$

Principle of optimality: The tail policy is optimal for the tail subproblem

The DP algorithm (stochastic case)

Intuition

- DP first solves ALL tail subproblems at the final stage
- At generic step, it solves ALL tail subproblems of a given time length, using solution of tail subproblems of shorter length

The DP algorithm (stochastic case)

The DP algorithm

- Start with

$$J_N(\boldsymbol{x}_N) = g_N(\boldsymbol{x}_N)$$

  and go backwards using

$$J_k(\boldsymbol{x}_k) = \min_{\boldsymbol{u}_k \in U(\boldsymbol{x}_k)} E_{w_k}\left[g_k(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k) + J_{k+1}\left(f(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k)\right)\right]$$

  for $k = 0, 1, \ldots, N-1$

- Then $J^*(\boldsymbol{x}_0) = J_0(\boldsymbol{x}_0)$ and optimal policy is constructed by setting

$$\pi_k^*(\boldsymbol{x}_k) = \underset{\boldsymbol{u}_k \in U(\boldsymbol{x}_k)}{\operatorname{argmin}} E_{w_k}\left[g_k(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k) + J_{k+1}\left(f(\boldsymbol{x}_k, \boldsymbol{u}_k, \boldsymbol{w}_k)\right)\right]$$

Example: Inventory Control Problem (1/4)

- Stock available $x_k \in \mathbb{N}$, inventory $u_k \in \mathbb{N}$, and demand $w_k \in \mathbb{N}$
- Dynamics: $x_{k+1} = \max(0, x_k + u_k - w_k)$
- Constraints: $x_k + u_k \leq 2$
- Probabilistic structure: $p(w_k = 0) = 0.1$, $p(w_k = 1) = 0.7$, and $p(w_k = 2) = 0.2$
- Cost

$$E\Big[\underbrace{0}_{g_3(x_3)} + \sum_{k=0}^{2} \underbrace{(u_k + (x_k + u_k - w_k)^2)}_{g_k(x_k, u_k, w_k)}\Big]$$

Example: Inventory Control Problem (2/4)

Example: Inventory Control Problem (3/4)

- Algorithm takes form

$$J_k(x_k) = \min_{0 \leq u_k \leq 2 - x_k} E_{w_k}[u_k + (x_k + u_k - w_k)^2 + J_{k+1}(\max(0, \; x_k + u_k - w_k))]$$

for $k = 0,1,2$

- For example

$$J_2(0) = \min_{u_2 = 0,1,2} E_{w_2}[u_2 + (u_2 - w_2)^2] = \min_{u_2 = 0,1,2} u_2 + 0.1(u_2)^2 + 0.7(u_2 - 1)^2 + 0.2(u_2 - 2)^2$$

which yields $J_2(0) = 1.3$, and $\pi_2^*(0) = 1$

Example: Inventory Control Problem (4/4)

Final solution:

- $J_0(0) = 3.7$,
- $J_0(1) = 2.7$, and
- $J_0(2) = 2.818$

Stochastic LQR

Find control policy that minimizes

$$E\left[\boldsymbol{x}_N^T Q \boldsymbol{x}_N + \sum_{k=0}^{N-1} \left(\boldsymbol{x}_k^T Q_k \boldsymbol{x}_k + \boldsymbol{u}_k^T R_k \boldsymbol{u}_k\right)\right]$$

subject to

- dynamics $\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + B_k \boldsymbol{u}_k + \boldsymbol{w}_k$

with $\boldsymbol{x}_0, \{\boldsymbol{w}_k\}$ independent and Gaussian vectors (and in addition $\{\boldsymbol{w}_k\}$ zero mean)

Stochastic LQR

Infinite Horizon MDPs

State: $x \in \mathcal{X}$ (often $s \in \mathcal{S}$)

Action: $u \in \mathcal{U}$ (often $a \in \mathcal{A}$)

Transition Function: $T(x_t \mid x_{t-1}, u_{t-1}) = p(x_t \mid x_{t-1}, u_{t-1})$

Reward Function: $r_t = R(x_t, u_t)$

Discount Factor: γ

MDP: $\mathcal{M} = (\mathcal{X}, \mathcal{U}, T, R, \gamma)$

Infinite Horizon MDPs

MDP: $\mathcal{M} = (\mathcal{X}, \mathcal{U}, T, R, \gamma)$

Stationary policy: $u_t = \pi(x_t)$

Goal: Choose policy that **maximizes cumulative reward**

$$\pi^* = \arg\max_{\pi} E\left[\sum_{t \geq 0} \gamma^t R\big(x_t, \pi(x_t)\big)\right]$$

Infinite Horizon MDPs

- The optimal reward $V^*(x)$ satisfies Bellman's equation

$$V^*(x) = \max_u \left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u)\, V^*(x') \right)$$

- For any stationary policy π, the reward $\mathrm{V}_\pi(x)$ is the unique solution to the equation

$$\mathrm{V}_\pi(x) = R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u)\, \mathrm{V}_\pi(x')$$

Solving infinite-horizon MDPs

If you know the model, use DP-ideas

- Value Iteration / Policy Iteration

RL: Learning from interaction

- Model-Based
- Model-free
  - Value based
  - Policy based

Value Iteration

- Initialize $V_0(x) = 0$ for all states x
- Loop until finite horizon / convergence:

$$V_{k+1}(x) = \max_u \left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x' | x, u) V_k(x') \right)$$

Q functions

$$V^*(x) = \max_u \underbrace{\left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u)\, V^*(x') \right)}$$

$$V^*(x) = \max_u Q^*(x,u)$$

- VI for Q functions

$$Q_{k+1}(x,u) = R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u) \max_u Q_k(x',u)$$

Policy Iteration

Suppose we have a policy $\pi_k(x)$

We can use VI to compute $V_{\pi_k}(x)$

Define $\pi_{k+1}(x) = \arg\max_u \left(R(x,u) + \gamma \sum_{x' \in \mathcal{X}} T(x'|x,u)\, V_{\pi_k}(x') \right)$

Proposition: $V_{\pi_{k+1}}(x) \geq V_{\pi_k}(x)\ \forall\, x \in \mathcal{X}$

Inequality is strict if π_k is suboptimal

Use this procedure to iteratively improve policy until convergence

Recap

- Value Iteration
  - Estimate optimal value function
  - Compute optimal policy from optimal value function

- Policy Iteration
  - Start with random policy
  - Iteratively improve it until convergence to optimal policy

- Require **model of MDP** to work!

Next time

- Belief space MDPs
- Dual control
- LQG
- Intro to reinforcement learning