

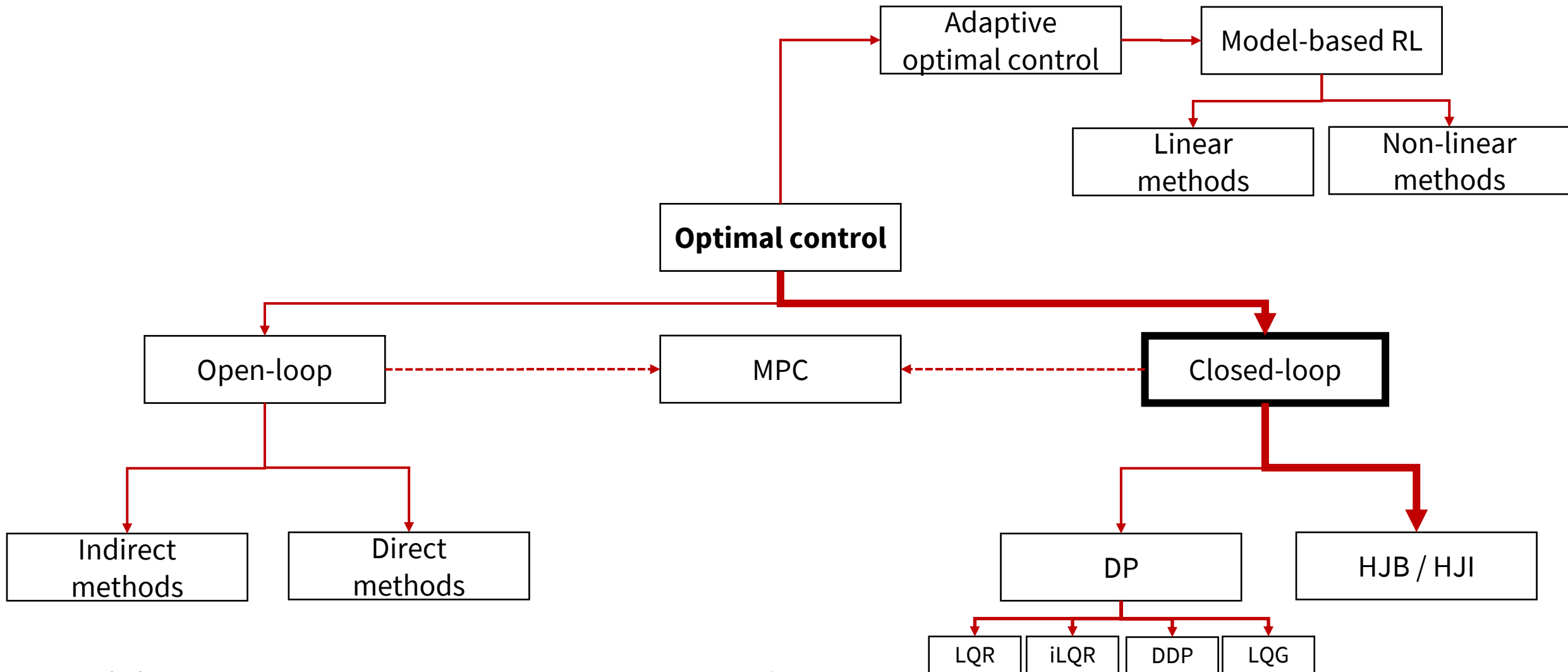
# AA203

# Optimal and Learning-based Control

HJB Equation and Continuous LQR



# Roadmap



# LQR extensions

Last week: focus on discrete-time setting

This week: focus on continuous-time setting

- dynamic programming approach leads to HJB equation: non-linear partial differential equation
- solution to continuous LQR problem
- differential games
- reachability analysis

# Continuous-time model

Last time:

- Model:  $\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, k)$ ,
- Cost:  $J(\mathbf{x}_0) = h_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g(\mathbf{x}_k, \mathbf{u}_k, k)$

This time:

- Model:  $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$ ,
- Cost:  $J(\mathbf{x}(0)) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau$

where  $t_0$  and  $t_f$  are fixed

# HJB equation

**Key idea:** apply principle of optimality

The “truncated” problem is


$$J(\mathbf{x}(t), \mathbf{u}(\tau)_{t \leq \tau \leq t_f}, t) = h(\mathbf{x}(t_f), t_f) + \int_t^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau$$

where  $t \leq t_f$  and  $\mathbf{x}(t)$  is an admissible state value

The optimal solution is

$$J^*(\mathbf{x}(t), t) = \min_{\mathbf{u}(\tau)_{t \leq \tau \leq t_f}} \left\{ h(\mathbf{x}(t_f), t_f) + \int_t^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \right\}$$

$J^*$  is the optimal cost-to-go at time  $t$  and state  $\mathbf{x}(t)$



# HJB equation

$$\begin{aligned} J^*(\mathbf{x}(t), t) &= \min_{\mathbf{u}(\tau)_{t \leq \tau \leq t_f}} \left\{ \int_t^{t+\Delta t} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau + h(\mathbf{x}(t_f), t_f) + \int_{t+\Delta t}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau \right\} \\ &= \min_{\mathbf{u}(\tau)_{t \leq \tau \leq t+\Delta t}} \left\{ \int_t^{t+\Delta t} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau + J^*(\mathbf{x}(t + \Delta t), t + \Delta t) \right\} \end{aligned}$$

where

- the second equality follows from the principle of optimality
- $J^*(\mathbf{x}(t + \Delta t), t + \Delta t)$  is the minimum cost of the process for the time interval  $t + \Delta t \leq \tau \leq t_f$  with initial state  $\mathbf{x}(t + \Delta t)$

# HJB equation

For small  $\Delta t$

$$J^*(\mathbf{x}(t), t) = \min_{\mathbf{u}(\tau)_{t \leq \tau \leq t + \Delta t}} \left\{ \int_t^{t + \Delta t} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau) d\tau + J^*(\mathbf{x}(t), t) \right. \\ \left. + \underbrace{\frac{\partial J^*}{\partial t}(\mathbf{x}(t), t)}_{:= J_t^*(\mathbf{x}(t), t)} \Delta t + \underbrace{\left[ \frac{\partial J^*}{\partial \mathbf{x}}(\mathbf{x}(t), t) \right]'}_{:= J_{\mathbf{x}}^*(\mathbf{x}(t), t)'} [\mathbf{x}(t + \Delta t) - \mathbf{x}(t)] + o(\Delta t) \right\}$$

where

$$\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$$

# HJB equation

Carrying this one step further

$$J^*(\mathbf{x}(t), t) = \min_{\mathbf{u}(t)} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) \Delta t + J^*(\mathbf{x}(t), t) \right. \\ \left. + J_t^*(\mathbf{x}(t), t) \Delta t + J_{\mathbf{x}}^*(\mathbf{x}(t), t)' [\mathbf{x}(t + \Delta t) - \mathbf{x}(t)] + o(\Delta t) \right\}$$

Hence, we obtain the equation

$$0 = J_t^*(\mathbf{x}(t), t) \Delta t + \min_{\mathbf{u}(t)} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) \Delta t \right. \\ \left. + J_{\mathbf{x}}^*(\mathbf{x}(t), t)' [\mathbf{x}(t + \Delta t) - \mathbf{x}(t)] + o(\Delta t) \right\}$$



# HJB equation

Dividing by  $\Delta t$ , accounting for control constraints, and taking  $\Delta t \rightarrow 0$ , we obtain the Hamilton-Jacobi-Bellman (HJB) equation

$$0 = J_t^*(\mathbf{x}(t), t) + \min_{\mathbf{u}(t) \in U} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + J_{\mathbf{x}}^*(\mathbf{x}(t), t)' \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \right\}$$

Note *constrained* optimization

with boundary condition

$$J^*(\mathbf{x}(t_f), t_f) = h(\mathbf{x}(t_f), t_f)$$

- Partial differential equation that should be satisfied for all state-time pairs  $(\mathbf{x}, t)$  by the cost-to-go functions  $J^*(\mathbf{x}(t), t)$
- Note: we assumed differentiability of  $J^*(\mathbf{x}(t), t)$

# Hamiltonian formalism

Define the Hamiltonian

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), J_{\mathbf{x}}^*, t) := g(\mathbf{x}(t), \mathbf{u}(t), t) + J_{\mathbf{x}}^*(\mathbf{x}(t), t)' \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

and

$$\mathcal{H}(\mathbf{x}(t), \mathbf{u}^*(\mathbf{x}(t), J_{\mathbf{x}}^*, t), J_{\mathbf{x}}^*, t) := \min_{\mathbf{u}(t) \in U} \mathcal{H}(\mathbf{x}(t), \mathbf{u}(t), J_{\mathbf{x}}^*, t)$$

Then, HJB equation can be written as

$$0 = J_t^*(\mathbf{x}(t), t) + \mathcal{H}(\mathbf{x}(t), \mathbf{u}^*(\mathbf{x}(t), J_{\mathbf{x}}^*, t), J_{\mathbf{x}}^*, t)$$

# Sufficiency theorem

Suppose  $V(\mathbf{x}, t)$  is a solution to the HJB equation, that is  $V$  is  $C^1$  in  $t$  and  $\mathbf{x}$ , and is such that

$$0 = V_t(\mathbf{x}, t) + \min_{\mathbf{u} \in U} \left\{ g(\mathbf{x}, \mathbf{u}, t) + V_{\mathbf{x}}(\mathbf{x}, t)' \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right\}$$

$$V(\mathbf{x}, t_f) = h(\mathbf{x}, t_f) \quad \text{for all } \mathbf{x}$$

Suppose also that  $\pi^*(\mathbf{x}, t)$  attains the minimum in this equation for all  $t$  and  $\mathbf{x}$ . Let  $\{\mathbf{x}^*(t) \mid t \in [t_0, t_f]\}$  be the state trajectory obtained from the given initial condition  $\mathbf{x}(0)$  when the control trajectory  $\mathbf{u}^*(t) = \pi^*(\mathbf{x}^*(t), t), t \in [t_0, t_f]$  is used. Then  $V$  is equal to the optimal cost-to-go function, i.e.,

$$V(\mathbf{x}, t) = J^*(\mathbf{x}, t) \quad \text{for all } \mathbf{x} \text{ and } t$$

Furthermore, the control trajectory  $\{\mathbf{u}^*(t) \mid t \in [t_0, t_f]\}$  is optimal.

# Example: continuous LQR

**Continuous LQR:** select control inputs to minimize

$$J(\mathbf{x}_0) = \frac{1}{2} \mathbf{x}(t_f)' H \mathbf{x}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [\mathbf{x}(t)' Q(t) \mathbf{x}(t) + \mathbf{u}(t)' R(t) \mathbf{u}(t)] dt$$

subject to the dynamics

$$\dot{\mathbf{x}}(t) = A(t) \mathbf{x}(t) + B(t) \mathbf{u}(t)$$

Assumptions:

- $H = H' \geq 0$ ,  $Q(t) = Q(t)' \geq 0$ ,  $R(t) = R(t)' > 0$
- $t_0$  and  $t_f$  specified
- $\mathbf{x}(t)$  and  $\mathbf{u}(t)$  unconstrained

# Example: continuous LQR

Consider the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \mathbf{x}(t)' Q(t) \mathbf{x}(t) + \frac{1}{2} \mathbf{u}(t)' R(t) \mathbf{u}(t) + J_{\mathbf{x}}^*(\mathbf{x}(t), t)' [A(t) \mathbf{x}(t) + B(t) \mathbf{u}(t)]$$

Necessary condition to minimize  $\mathbf{u}(t)$  is that  $\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = 0$ , that is:

$$\frac{\partial \mathcal{H}}{\partial \mathbf{u}} = R(t) \mathbf{u}(t) + B(t)' J_{\mathbf{x}}^*(\mathbf{x}(t), t) = 0$$

Since  $\frac{\partial^2 \mathcal{H}}{\partial \mathbf{u}^2} = R(t) > 0$ , the control that satisfies the NOC is a global minimum; solving:

$$\mathbf{u}^*(t) = -R^{-1}(t) B(t)' J_{\mathbf{x}}^*(\mathbf{x}(t), t)$$

# Example: continuous LQR

Plugging in:

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\mathbf{x}(t)'Q(t)\mathbf{x}(t) + \frac{1}{2}J_{\mathbf{x}}^*(\mathbf{x}(t), t)'B(t)R(t)^{-1}B(t)'J_{\mathbf{x}}^*(\mathbf{x}(t), t) \\ &\quad + J_{\mathbf{x}}^*(\mathbf{x}(t), t)'A(t)\mathbf{x}(t) - J_{\mathbf{x}}^*(\mathbf{x}(t), t)'B(t)R(t)^{-1}B(t)'J_{\mathbf{x}}^*(\mathbf{x}(t), t) \\ &= \frac{1}{2}\mathbf{x}(t)'Q(t)\mathbf{x}(t) - \frac{1}{2}J_{\mathbf{x}}^*(\mathbf{x}(t), t)'B(t)R(t)^{-1}B(t)'J_{\mathbf{x}}^*(\mathbf{x}(t), t) \\ &\quad + J_{\mathbf{x}}^*(\mathbf{x}(t), t)'A(t)\mathbf{x}(t)\end{aligned}$$

# Example: continuous LQR

The HJB equation is then

$$0 = J_t^*(\mathbf{x}(t), t) + \frac{1}{2}\mathbf{x}(t)'Q(t)\mathbf{x}(t) \\ - \frac{1}{2}J_{\mathbf{x}}^*(\mathbf{x}(t), t)'B(t)R(t)^{-1}B(t)'J_{\mathbf{x}}^*(\mathbf{x}(t), t) + J_{\mathbf{x}}^*(\mathbf{x}(t), t)'A(t)\mathbf{x}(t)$$

with boundary conditions

$$J^*(\mathbf{x}(t_f), t_f) = \frac{1}{2}\mathbf{x}(t_f)'H\mathbf{x}(t_f)$$

# Example: continuous LQR

Ansatz (lucky guess):

$$J^*(\mathbf{x}(t), t) = \frac{1}{2} \mathbf{x}(t)' K(t) \mathbf{x}(t) \quad \text{for some } K(t) > 0$$

Substituting

$$\begin{aligned} 0 = & \frac{1}{2} \mathbf{x}(t)' \dot{K}(t) \mathbf{x}(t) + \frac{1}{2} \mathbf{x}(t)' Q(t) \mathbf{x}(t) \\ & - \frac{1}{2} \mathbf{x}(t)' K(t) B(t) R(t)^{-1} B(t)' K(t) \mathbf{x}(t) + \mathbf{x}(t)' K(t) A(t) \mathbf{x}(t) \end{aligned}$$

Note:

- $J_{\mathbf{x}}^* = K(t) \mathbf{x}(t)$  and  $J_t^* = \frac{1}{2} \mathbf{x}(t)' \dot{K}(t) \mathbf{x}(t)$
- $\mathbf{x}(t)' K(t) A(t) \mathbf{x}(t) = \frac{1}{2} \mathbf{x}(t)' K(t) A(t) \mathbf{x}(t) + \frac{1}{2} \mathbf{x}(t)' A(t)' K(t) \mathbf{x}(t)$



# Example: continuous LQR

So we obtain

$$0 = \frac{1}{2} \mathbf{x}(t)' \left( \dot{K}(t) + Q(t) - K(t)B(t)R(t)^{-1}B(t)'K(t) + K(t)A(t) + A(t)'K(t) \right) \mathbf{x}(t)$$

Since this equation must hold for all  $\mathbf{x}(t)$ , we obtain

$$-\dot{K}(t) = Q(t) - K(t)B(t)R(t)^{-1}B(t)'K(t) + K(t)A(t) + A(t)'K(t)$$

with boundary condition  $K(t_f) = H$

# Example: continuous LQR

- The HJB equation reduces to a set of differential equation (the Riccati equation)
- The Riccati equation is integrated **backwards**
- Once we find  $K(t)$ , the control policy is

$$\mathbf{u}^*(t) = -R(t)^{-1} B(t)' K(t) \mathbf{x}(t)$$

- Analogously to the discrete case, under some additional assumptions,  $K(t) \rightarrow \text{constant}$  in the infinite horizon setting

# Next time

- HJI and reachability analysis