Optimal and Learning-Based Control

Pontryagin’s maximum principle and indirect methods

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1. Geometry and generalizations of first-order NOCs

2. Weak Pontryagin maximum principle in discrete-time

3. Weak Pontryagin maximum principle in continuous-time

4. Pontryagin maximum principle in continuous-time

5. Indirect methods for optimal control

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1. Geometry and generalizations of first-order NOCs

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Review: First-order NOCs

\[
\begin{align*}
\text{minimize } & f(x) \\
\text{subject to } & h(x) = 0 \\
& g(x) \preceq 0
\end{align*}
\]

\[L(x, \lambda, \mu) := f(x) + \lambda^T h(x) + \mu^T g(x)\]

**Theorem (First-order NOCs)**

Suppose \( x^* \in \mathbb{R}^n \) is a local minimum of \( f \in C^1(\mathbb{R}^n, \mathbb{R}) \) subject to \( h(x^*) = 0 \) and \( g(x^*) \preceq 0 \) with \( h \in C^1(\mathbb{R}^n, \mathbb{R}^m) \) and \( g \in C^1(\mathbb{R}^n, \mathbb{R}^r) \). Moreover, assume

\[
\{\nabla h_i(x^*)\}_{i=1}^m \cup \{\nabla g_j(x^*)\}_{j \in A_g(x^*)}
\]

are linearly independent. Then there exist unique \( \lambda^* \in \mathbb{R}^m \) and \( \mu^* \in \mathbb{R}^r \) such that

\[
\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \quad \mu^* \succeq 0, \quad \mu_j^* = 0, \ \forall j \notin A_g(x^*),
\]

The assumption on the constraint gradients is known as the *linear independence constraint qualification (LICQ)*.
Geometry of first-order NOCs

Tangent cone $\mathcal{T}_X(x)$ “vectors that stay in $\mathcal{X}$”

Normal cone $\mathcal{N}_X(x)$ “vectors that leave $\mathcal{X}$”

If $x^*$ is a local minimum of $f$ over $\mathcal{X}$, then

$-\nabla f(x^*) \in \mathcal{N}_X(x^*)$, i.e., there is no feasible component of $-\nabla f(x^*)$ that would allow us to locally decrease $f(x^*)$.

For convenience, we write “$-\nabla f(x^*) \perp x^* \mathcal{X}$”. In other literature, you may see “$-\nabla f(x^*) \perp \mathcal{T}_X(x^*)$”.

If $\mathcal{X} = \{x \in \mathbb{R}^n \mid h(x) = 0, g(x) \preceq 0\}$ and the LICQ holds at $x^* \in \mathcal{X}$, then

$$\mathcal{T}_X(x^*) = \left\{ d \in \mathbb{R}^n \mid \frac{\partial h}{\partial x}(x^*) d = 0, \nabla g_j(x^*)^T d \leq 0, \forall j \in A_g(x^*) \right\}$$

$$\mathcal{N}_X(x^*) = \left\{ v \in \mathbb{R}^n \mid v = \frac{\partial h}{\partial x}(x^*)^T \lambda + \frac{\partial g}{\partial x}(x^*)^T \mu, \mu \succeq 0, \mu_j = 0, \forall j \notin A_g(x^*) \right\}$$
Example: A problem with linearly dependent constraints

minimize \( f(x) := x_1 + x_2 \)
subject to \( h_1(x) := (x_1 - 1)^2 + x_2^2 - 1 = 0 \)
\( h_2(x) := (x_1 - 2)^2 + x_2^2 - 4 = 0 \)

At the only feasible point \( x^* = 0 \), we have
\[
\nabla f(x^*) = (1, 1) \\
\nabla h_1(x^*) = (-2, 0), \quad \nabla h_2(x^*) = (-4, 0)
\]

The constraint gradients are linearly dependent (i.e., the LICQ does not hold), so we cannot write \( \nabla f(x^*) + \lambda_1^* \nabla h_1(x^*) + \lambda_2^* \nabla h_2(x^*) = 0 \).

In essence, the constraints “pinch together” so that just one \( x^* \) is feasible, regardless of the objective value.
Theorem (Fritz John first-order NOCs)

Let \( f \in C^1(\mathbb{R}^n, \mathbb{R}) \), \( h \in C^1(\mathbb{R}^n, \mathbb{R}^m) \), and \( g \in C^1(\mathbb{R}^n, \mathbb{R}^r) \). Suppose \( x^* \in \mathbb{R}^n \) is a local minimum of the problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad h(x) = 0 \cdot \\
& \quad g(x) \preceq 0
\end{align*}
\]

Then there exist \((\eta, \lambda^*, \mu^*) \in \{0, 1\} \times \mathbb{R}^m \times \mathbb{R}^r\) such that

\[
(\eta, \lambda^*, \mu^*) \neq 0 \quad \text{non-triviality}
\]

\[
- \nabla_x L_\eta(x^*, \lambda^*, \mu^*) \perp_{x^*} S \quad \text{stationarity}
\]

\[
\mu_j^* \geq 0, \quad \mu_j^* g_j(x^*) = 0, \quad \forall j \in \{1, 2, \ldots, r\} \quad \text{complementarity}
\]

where \( L_\eta(x, \lambda, \mu) \) is the partial Lagrangian

\[
L_\eta(x, \lambda, \mu) := \eta f(x) + \lambda^T h(x) + \mu^T g(x).
\]
Theorem (Fritz John first-order NOCs)

If $x^*$ is a local minimum, there exist $(\eta, \lambda^*, \mu^*) \in \{0, 1\} \times \mathbb{R}^m \times \mathbb{R}^r$ such that

$$(\eta, \lambda^*, \mu^*) \neq 0$$

$$- \nabla_x L_{\eta}(x^*, \lambda^*, \mu^*) \perp_{x^*} S$$

$$\mu_j^* \geq 0, \mu_j^* g_j(x^*) = 0, \forall j \in \{1, 2, \ldots, r\}$$

where $L_{\eta}(x, \lambda, \mu)$ is the partial Lagrangian

$L_{\eta}(x, \lambda, \mu) := \eta f(x) + \lambda^T h(x) + \mu^T g(x)$.

The “abnormal case” $\eta = 0$ yields necessary conditions independent of the objective $f$.

Corollary

If $S = \mathbb{R}^n$ and the LICQ holds, then $\eta = 1$ and $\nabla_x L_1(x^*, \lambda^*, \mu^*) = 0$. 
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6. Time-optimal control problems
Consider the discrete-time optimal control problem (OCP)

\[
\begin{align*}
\text{minimize} & \quad \ell_T(x_T) + \sum_{t=0}^{T-1} \ell(t, x_t, u_t) \\
\text{subject to} & \quad x_{t+1} = f(t, x_t, u_t), \quad \forall t \in \{0, 1, \ldots, T-1\} \\
& \quad x_0 = \bar{x}_0 \\
& \quad x_T \in X_T \\
& \quad u_t \in U, \quad \forall t \in \{0, 1, \ldots, T-1\}
\end{align*}
\]

cost (terminal + stage)
dynamical feasibility
initial condition
terminal condition
input constraints

An optimal control \( u^* = \{u_t^*\}_{t=0}^{T-1} \) for a specific initial state \( \bar{x}_0 \) is an open-loop input.

An optimal control of the form \( u_t^* = \pi^*(t, x_t) \) is a closed-loop input.
The partial Lagrangian is

\[ L_\eta(x, u, p) = \eta \ell_T(x_T) + p_0^T(x_0 - \bar{x}_0) + \sum_{t=0}^{T-1} \left( \eta \ell(t, x_t, u_t) + p_{t+1}^T(x_{t+1} - f(t, x_t, u_t)) \right) \]

\[ = \eta \ell_T(x_T) + p_0^T(x_0 - \bar{x}_0) + \sum_{t=0}^{T-1} (p_{t+1}^T x_{t+1} - H_\eta(t, x_t, u_t, p_{t+1})) \]

with normality \( \eta \in \{0, 1\} \), Lagrange multipliers \( \{p_t\}_{t=0}^T \subset \mathbb{R}^n \), and Hamiltonian

\[ H_\eta(t, x, u, p) := p^T f(t, x, u) - \eta \ell(t, x, u). \]

Setting \( \nabla_{x_t} L(x^*, u^*) = 0 \) for \( t \in \{0, 1, \ldots, T - 1\} \) yields

\[ p_t^* = \nabla_x H_\eta(t, x_t^*, u_t^*, p_{t+1}^*), \quad \forall t \in \{0, 1, \ldots, T - 1\}, \]

which is a backwards recursion for the adjoint or co-state \( p_t^* \).
Transversality and the maximum condition (discrete-time)

The partial Lagrangian is

\[ L_\eta(x, u, p) = \eta\ell_T(x_T) + p_0^T(x_0 - \bar{x}_0) + \sum_{t=0}^{T-1} (p_{t+1}^Tx_{t+1} - H_\eta(t, x_t, u_t, p_{t+1})) \]

where we left out \( x_T \in \mathcal{X}_T \) and \( u_t \in \mathcal{U} \). Setting \(-\nabla_{x_T} L_\eta(x^*, u^*) \perp x_T^* \mathcal{X}_T\) yields the transversality condition

\[ -p_{T}^* - \eta \nabla\ell_T(x_T^*) \perp x_T^* \mathcal{X}_T, \]

and setting \(-\nabla_{u_t} L(x^*, u^*) \perp u_t^* \mathcal{U}\) yields the weak maximum condition

\[ \nabla_u H_\eta(t, x_t^*, u_t^*, p_{t+1}^*) \perp u_t^* \mathcal{U}, \ \forall t \in \{0, 1, \ldots, T - 1\}. \]

We refer to this condition as “weak” since it is a necessary, but not sufficient condition for a solution of the problem

\[ \max_{u \in \mathcal{U}} H_\eta(t, x_t^*, u, p_{t+1}^*). \]
Collect these necessary conditions together to get the Pontryagin maximum principle (PMP).

**Theorem (Pontryagin maximum principle (discrete-time))**

Let \((x^*, u^*)\) be a local minimum of the discrete-time OCP with terminal set \(X_T\) and control set \(U\). Then \(\eta \in \{0, 1\}\) and \(\{p_t^*\}_{t=0}^T \subset \mathbb{R}^n\) exist such that

\[
(\eta, p_0^*, p_1^*, \ldots, p_T^*) \neq 0 \quad \text{non-triviality}
\]

\[
p_t^* = \nabla_x H_\eta(t, x_t^*, u_t^*, p_{t+1}^*) , \quad \forall t \in \{0, 1, \ldots, T-1\} \quad \text{adjoint equation}
\]

\[
-p_T^* - \eta \nabla \ell_T(x_T^*) \perp x_T^* X_T \quad \text{transversality}
\]

\[
\nabla_u H_\eta(t, x_t^*, u_t^*, p_{t+1}^*) \perp u_t^* U , \quad \forall t \in \{0, 1, \ldots, T-1\} \quad \text{maximum condition (weak)}
\]
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6. Time-optimal control problems
Consider the continuous-time optimal control problem (OCP)

\[
\begin{align*}
\text{minimize} & \quad \ell_T(x(T)) + \int_0^T \ell(t, x(t), u(t)) \, dt \\
\text{subject to} & \quad \dot{x}(t) = f(t, x(t), u(t)), \quad \forall t \in [0, T] \\
& \quad x(0) = x_0 \quad \text{initial condition} \\
& \quad x(T) \in \mathcal{X}_T \quad \text{terminal condition} \\
& \quad u(t) \in \mathcal{U}, \quad \forall t \in [0, T] \quad \text{input constraints}
\end{align*}
\]

An optimal control \( u^*(t) \) for a specific initial state \( x_0 \) is an \textit{open-loop} input.

An optimal control of the form \( u^*(t) = \pi^*(t, x(t)) \) is a \textit{closed-loop} input.
Consider piecewise continuous trajectories such that \( x(t) = x(t_k) \) and \( u(t) = u(t_k) \) for \( t \in [t_k, t_{k+1}) \), with \( k \in \{0, 1, \ldots, N - 1\} \), \( t_0 = 0 \) and \( t_N = T \).

Define \( \Delta t_k := t_{k+1} - t_k \) such that \( \Delta t_k > 0 \) for all \( k \in \{0, 1, \ldots, N - 1\} \).

Consider the discretized OCP

\[
\begin{align*}
\text{minimize} & \quad \ell_T(x(t_N)) + \sum_{k=0}^{N-1} \Delta t_k \ell(t_k, x(t), u(t)) \\
\text{subject to} & \quad x(t_{k+1}) = x(t_k) + \Delta t_k f(t_k, x(t), u(t)), \quad \forall k \in \{0, 1, \ldots, N - 1\} \\
& \quad x(t_0) = x_0 \\
& \quad x(t_N) \in \mathcal{X}_T \\
& \quad u(t_k) \in \mathcal{U}, \quad \forall k \in \{0, 1, \ldots, N - 1\}
\end{align*}
\]
Use the discrete-time PMP on a local minimum \((x^*, u^*)\) of the discretized OCP to get

\[
(\eta, p(t_0), p(t_1), \ldots, p(t_N)) \neq 0
\]

\[
- \frac{(p^*(t_{k+1}) - p^*(t_k))}{\Delta t_k} = \nabla_x H_\eta(t_k, x^*(t_k), u^*(t_k), p^*(t_{k+1})), \quad \forall k \in \{0, 1, \ldots, N - 1\}
\]

\[
- p^*(t_N) - \eta \nabla \ell_T(x^*(t_N)) \perp_{x^*(t_N)} X_T
\]

\[
\nabla_u H_\eta(t_k, x^*(t_k), u^*(t_k), p^*(t_{k+1})) \perp_{u^*} U, \quad \forall k \in \{0, 1, \ldots, N - 1\}
\]

where we use the *continuous-time Hamiltonian*

\[
H_\eta(t, x, u, p) := p^T f(t, x, u) - \eta \ell(t, x, u).
\]
The above conditions suggest the following continuous-time PMP as $\Delta t_k \to 0$.

**Theorem (Pontryagin maximum principle (continuous-time, weak))**

Let $(x^*, u^*)$ be a local minimum of the continuous-time optimal control problem with terminal set $\mathcal{X}_T$ and control set $\mathcal{U}$. Then $\eta \in \{0, 1\}$ and $p^* : [0, T] \to \mathbb{R}^n$ exist such that

\[
\begin{align*}
(\eta, p(t)) &\neq 0 \quad \text{non-triviality} \\
-\dot{p}^*(t) &= \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \quad \forall t \in [0, T] \quad \text{adjoint equation} \\
-p^*(T) - \eta \nabla \ell_T(x^*(T)) &\perp_{x^*(T)} \mathcal{X}_T \quad \text{transversality} \\

- \nabla H_\eta(t, x^*(t), u^*(t), p^*(t)) &\perp_{u^*(t)} \mathcal{U}, \quad \forall t \in [0, T] \quad \text{maximum condition (weak)}
\end{align*}
\]

“(\eta, p(t)) \neq 0” means there exists at least one $t \in [0, T]$ such that $(\eta, p(t)) \neq 0$. 
Agenda

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Recall that \((x^*, u^*)\) is a local minimum of \(J(x^*, u^*)\) if there exists \(\varepsilon > 0\) such that \(J(x^*, u^*) \leq J(x, u)\) for all \((x, u)\) in the \(\varepsilon\)-sized norm ball around \((x^*, u^*)\).

In using the discrete-time PMP as a heuristic to obtain the continuous-time PMP, we are implicitly using the \(C^0\)-norm for both \(x^*\) and \(u^*\), i.e.,

\[
\|x - x^*\|_{C^0} := \max_{t \in [0, T]} \|x(t) - x^*(t)\|, \quad \|u - u^*\|_{C^0} := \max_{t \in [0, T]} \|u(t) - u^*(t)\|.
\]

We can strengthen the continuous-time PMP if we use the \(C^0\)-norm for \(x^*\) and the \(L^1\)-norm for \(u^*\), i.e.,

\[
\|x - x^*\|_{C^0} := \max_{t \in [0, T]} \|x(t) - x^*(t)\|, \quad \|u - u^*\|_{L^1} := \int_0^T \|u(t) - u^*(t)\| \, dt.
\]
Strengthening the maximum condition via needle perturbations

In general, the $L^1$-norm ball for $u^*$ allows for large pointwise variations at each time $t$. Suppose the control set $\mathcal{U}$ is bounded, i.e., $\|u - v\| \leq c$ for all $u, v \in \mathcal{U}$ and some $c > 0$.

Given some $u^* : [0, T] \rightarrow \mathcal{U}$, any $\tau \in [0, T)$ and $\varepsilon > 0$ such that $[\tau, \tau + \varepsilon) \subset [0, T]$, and any $v \in \mathcal{U}$, define

$$u(t) = \begin{cases} v, & t \in [\tau, \tau + \varepsilon) \\ u^*(t), & t \in [0, \tau) \cup [\tau + \varepsilon, T] \end{cases}$$

This is a spatial needle perturbation of $u^*(t)$. Then it can be shown that

$$\|u - u^*\|_{L^1} := \int_0^T \|u(t) - u^*(t)\| \, dt = \int_{\tau}^{\tau + \varepsilon} \|v - u^*(t)\| \, dt \leq \int_{\tau}^{\tau + \varepsilon} c \, dt = \varepsilon c.$$

$$x(T) \approx x^*(T) + \varepsilon d, \quad d \in \mathcal{T}_x(T)(x^*(T))$$

for small enough $\varepsilon$. Overall, a large spatial perturbation in $u^*(t)$ can correspond to small feasible perturbations to both $x^*$ and $u^*$. 

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The possibility of large spatial control perturbations still corresponding to “feasible neighbours” of \((x^*, u^*)\) suggests the following strengthened PMP.

**Theorem (Pontryagin maximum principle (continuous-time))**

Let \((x^*, u^*)\) be a local minimum (using the \(C^0\)-norm and \(L^1\)-norm, respectively) of the continuous-time OCP with terminal set \(X_T\) and bounded control set \(U\). Then \(\eta \in \{0, 1\}\) and \(p^* : [0, T] \to \mathbb{R}^n\) exist such that

\[
(\eta, p^*(t)) \neq 0 \quad \text{non-triviality}
\]

\[
-p^*(t) = \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \ \forall t \in [0, T] \quad \text{adjoint equation}
\]

\[
-p^*(T) - \eta \nabla \ell_T(x^*(T)) \perp x^*(T) X_T \quad \text{transversality}
\]

\[
H_\eta(t, x^*(t), u^*(t), p^*(t)) = \sup_{u \in U} H_\eta(t, x^*(t), u, p^*(t)), \ \forall t \in [0, T] \quad \text{maximum condition}
\]

A rigorous proof relies on variational calculus (Liberzon, 2012; Clarke, 2013).
Consider the continuous-time OCP

\[
\begin{align*}
\text{minimize} & \quad \int_0^T \sum_{j=1}^m \alpha_j |u_j(t)| \, dt \\
\text{subject to} & \quad \dot{x}(t) = a(t, x(t)) + \sum_{j=1}^m u_j(t)b_j(t, x(t)), \ \forall t \in [0, T] \\
& \quad x(0) = x_0 \\
& \quad x(T) = 0 \\
& \quad -\bar{u} \leq u(t) \leq \bar{u}, \ \forall t \in [0, T]
\end{align*}
\]

where \( \bar{u} > 0 \). The Hamiltonian is

\[
H_\eta(t, x, u, p) = p^T \left( a(t, x) + \sum_{j=1}^m u_j b_j(t, x) \right) - \eta \sum_{j=1}^m \alpha_j |u_j|
\]
Example: Minimum fuel for a control-affine system

The Hamiltonian is

\[ H_\eta(t, x, u, p) = a(t, x)^T p + \sum_{j=1}^{m} (u_j b_j(t, x)^T p - \eta \alpha_j |u_j|) \]

The adjoint equation is

\[ \dot{p}^* = -\nabla_x H_\eta(t, x^*, u^*, p^*) = -\frac{\partial a}{\partial x}(t, x^*)p^* - \sum_{j=1}^{m} u_j^* \frac{\partial b_j}{\partial x}(t, x^*)p^* \]

The maximum condition is

\[ u_j^* = \arg\max_{u_j \in [-\bar{u}_j, \bar{u}_j]} (u_j b_j(t, x^*)^T p^* - \eta \alpha_j |u_j|) = \begin{cases} -\bar{u}_j, & b_j(t, x^*)^T p^* < -\eta \alpha_j \\ 0, & b_j(t, x^*)^T p^* \in [-\eta \alpha_j, \eta \alpha_j] \\ \bar{u}_j, & b_j(t, x^*)^T p^* > \eta \alpha_j \end{cases} \]

which for \( \eta = 1 \) is an example of “bang-off-bang” control.
Assume $\eta = 1$, i.e., the “normal” case. Altogether, we have the boundary value problem (BVP)

\[
\begin{align*}
(\dot{x}^*) &= \left(\begin{array}{c}
a(t, x^*) + \sum_{j=1}^{m} u_j^* b_j(t, x^*) \\
-\frac{\partial a}{\partial x}(t, x^*) p^* - \sum_{j=1}^{m} u_j^* \frac{\partial b_j}{\partial x}(t, x^*) p^*
\end{array}\right), \\
u_j^* &= \begin{cases} 
-\bar{u}_j, & b_j(t, x^*)^T p^* < -\alpha_j \\
0, & b_j(t, x^*)^T p^* \in [-\alpha_j, \alpha_j] \\
\bar{u}_j, & b_j(t, x^*)^T p^* > \alpha_j
\end{cases}
\end{align*}
\]

with boundary conditions $x^*(0) = x_0$ and $x^*(T) = 0$.

Transversality did not factor into this problem, since the normal cone of the singleton $X_T = \{0\}$ is just $\mathbb{R}^n$ (i.e., any direction “leaves” the terminal set).
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An *indirect method* generally focuses on solving the BVP

\[
\begin{pmatrix}
\dot{x}^* \\
\dot{p}^*
\end{pmatrix} = \begin{pmatrix}
f(t, x^*, u^*) \\
-\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*)
\end{pmatrix}, \quad x^*(0) = x_0, \quad h(x^*(T), p^*(T)) = 0.
\]

where \(h(x^*(T), p^*(T)) \in \mathbb{R}^n\). The *open-loop* optimal control candidate \(u^*(t, x^*(t), p^*(t))\) is then extracted.

The boundary condition \(h(x^*(T), p^*(T)) = 0\) is determined by the terminal set constraint \(x^*(T) \in \mathcal{X}_T\) and the transversality condition \(-p^*(T) - \eta \nabla \ell_T(x^*(T)) \perp_{x^*(T)} \mathcal{X}_T\).

We are implicitly assuming an optimal control exists. Even then, there may be multiple local optima.
Shooting methods

To solve the BVP

\[
\left( \begin{array}{c} \dot{x}^* \\ \dot{p}^* \end{array} \right) = \left( \begin{array}{c} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{array} \right), \quad x^*(0) = x_0, \quad h(x^*(T), p^*(T)) = 0,
\]

we consider the associated initial value problem (IVP)

\[
\left( \begin{array}{c} \dot{x}^* \\ \dot{p}^* \end{array} \right) = \left( \begin{array}{c} f(t, x^*, u^*) \\ -\nabla_x H_\eta(t, x^*, u^*(t, x^*, p^*), p^*) \end{array} \right), \quad x^*(0) = x_0, \quad p^*(0) = p_0.
\]

We can integrate the IVP forward in time to get \( x^*(T; p_0) \) and \( p^*(T; p_0) \), which are parameterized by \( p_0 \).

We can use a root-finding method (e.g., bisection search, Newton-Raphson method) to find \( p_0 \) such that \( h(x^*(T; p_0), p^*(T; p_0)) = 0 \). This is called single shooting and gives us a solution of the BVP.
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Consider the continuous-time OCP

\[
\begin{align*}
\text{minimize} \quad & \ell_T(T, x(T)) + \int_0^T \ell(t, x(t), u(t)) \, dt \\
\text{subject to} \quad & \dot{x}(t) = f(t, x(t), u(t)), \quad \forall t \in [0, T] \\
& x(0) = x_0 \\
& x(T) \in X_T \\
& u(t) \in \mathcal{U}, \quad \forall t \in [0, T]
\end{align*}
\]

The final time \( T \) is now a free variable (subject to \( T \geq 0 \)).
Use the change of variables $t(s) = Ts$ with $s \in [0, 1]$ to get

$\min_{(x,t),(u,T)} \ell_T(t(1), x(1)) + T \int_0^1 \ell(t(s), x(s), u(s)) \, ds$

subject to $\dot{x}(s) = T f(t(s), x(s), u(s)), \quad \dot{t}(s) = T, \quad \forall s \in [0, 1]$

$x(0) = x_0, \quad t(0) = 0$

$x(1) \in \mathcal{X}_T$

$u(s) \in \mathcal{U}, \quad T \in [0, \infty), \quad \forall s \in [0, 1]$

To derive a new form of the PMP for time-optimal problems, we apply the fixed final time PMP to the problem above, where we treat $t$ and $T$ as a new state and input, respectively.
Deriving the time-optimal PMP

Applying the fixed final time PMP gives us the Hamiltonian

\[ \tilde{H}_\eta(s, x, t, u, T, p, \lambda) = T(H(t, x, u, p) + \lambda), \]

where \( H(t, x, u, p) \) is the usual Hamiltonian, and \( \lambda \) is the adjoint for the new "state" \( t(s) = Ts \). Taking derivatives with respect to \((x, t)\) yields the adjoint equations

\[
\frac{dp^*}{ds} = -T^* \nabla_x H(t, x^*, u^*, p^*), \quad \frac{d\lambda^*}{ds} = -T^* \frac{\partial H}{\partial t}(t, x^*, u^*, p^*),
\]

which by the chain rule with \( \frac{dt}{ds} = T \) become

\[
\dot{p}^* = -\nabla_x H(t, x^*, u^*, p^*), \quad \dot{\lambda}^* = -\frac{\partial H}{\partial t}(t, x^*, u^*, p^*).
\]

Since \( t \) has no terminal constraint, we have the transversality conditions

\[
-p^*(1) - \eta \nabla_x \ell_T(t(1), x^*(1)) \perp_{x^*(1)} X_T, \quad -\lambda^*(1) - \eta \nabla_T \ell_T(t(1), x^*(1)) = 0.
\]

which after using \( t = sT \) gives us

\[
-p^*(T) - \eta \nabla_x \ell_T(T^*, x^*(T)) \perp_{x^*(T)} X_T, \quad -\lambda^*(T^*) = \eta \nabla_T \ell_T(T^*, x^*(T)).
\]
Applying the fixed final time PMP gives us the Hamiltonian

\[ \tilde{H}_\eta(s, x, t, u, T, p, \lambda) = T(H(t, x, u, p) + \lambda), \]

where \( H(t, x, u, p) \) is the usual Hamiltonian, and \( \lambda \) is the adjoint for the new "state" \( t(s) = Ts \).

We are considering the absolute value norm for \( T \), and \([0, \infty)\) is unbounded. So we use the maximum condition for \( u^* \) and the weak maximum condition for \( T^* \) to get

\[ \nabla_T \tilde{H}_\eta(t, x^*, u^*, p^*) \perp_{T^*} [0, \infty) \implies H(t, x^*, u^*, p^*) + \lambda^* = 0, \]

where we have assumed \( T^* > 0 \) to get that the normal cone is just \( \{0\} \). Evaluating this condition at \( t = T^* \) gives us

\[ H(T^*, x^*(T^*), u^*(T^*), p^*(T^*)) = -\lambda^*(T^*) = \eta \nabla_t \ell_T(T^*, x^*(T)), \]

which is the additional boundary condition we need for free final time \( T^* \).
Collecting all of the conditions we derived above gives us the free final time PMP.

**Theorem (Pontryagin maximum principle (continuous-time, free final time))**

Let \((x^*, u^*, T^*)\) be a local minimum (using the \(C^0\)-norm, \(L^1\)-norm, and absolute value, respectively) of the continuous-time OCP with terminal set \(X_T\), bounded control set \(U\), and free final time \(T \geq 0\). Then \(\eta \in \{0, 1\}\) and \(p^*: [0, T^*] \to \mathbb{R}^n\) exist such that

\[
(\eta, p^*(t)) \neq 0 \quad \text{non-triviality}
\]

\[-p^*(t) = \nabla_x H_\eta(t, x^*(t), u^*(t), p^*(t)), \ \forall t \in [0, T^*] \quad \text{adjoint equation}
\]

\[-p^*(T^*) - \eta \nabla \ell_T(T^*, x^*(T^*)) \perp_{x^*(T)} X_T \quad \text{transversality}
\]

\[H_\eta(t, x^*(t), u^*(t), p^*(t)) = \sup_{u \in U} H_\eta(t, x^*(t), u, p^*(t)), \ \forall t \in [0, T^*] \quad \text{maximum condition}
\]

\[H_\eta(T^*, x^*(T^*), u^*(T^*), p^*(T^*)) = \eta \frac{\partial \ell_T}{\partial T}(T^*, x^*(T^*)) \quad \text{maximum condition (boundary)}
\]
Direct methods for optimal control
(i.e., solving discretized optimal control problems directly)