AA203
Optimal and Learning-based Control

Direct methods for optimal control: fundamental concepts*

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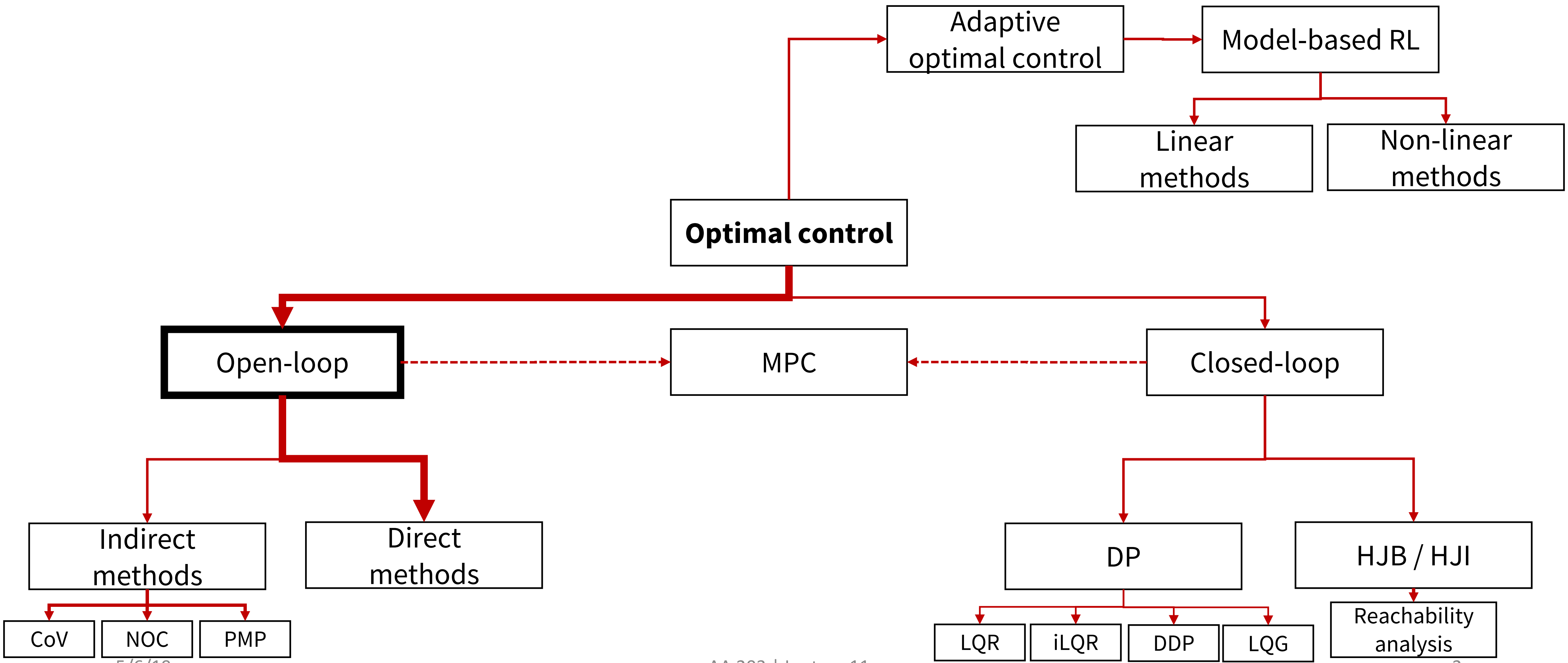
Roadmap



Adaptive optimal control
Model-based RL
Linear methods
Non-linear methods
Optimal control
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI
CoV
NOC
PMP
LQR
iLQR
DDP
LQG
Reachability analysis

Optimal control problem

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

(**OCP**)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),\ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0,\ \ \mathbf{x}(t_f) \in M_f = \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]$$

For simplicity:

- We assume the terminal cost h is equal to 0
- We assume $t_0 = 0$

- Indirect Methods:
  1. Apply necessary conditions for optimality to (**OCP**)
  2. Solve a two-point boundary value problem

- Direct Methods:
  1. Transcribe (**OCP**) into a nonlinear, constrained optimization problem
  2. Solve the optimization problem via nonlinear programming

Transcription into nonlinear programming

(OCP)

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \;\; t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0, \;\; \mathbf{x}(t_f) \in M_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \;\; t \in [0, t_f]$$

Forward Euler time discretization

1. Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and, for every $i = 0, \dots, N-1$, define $\mathbf{x}_{i+1} \sim \mathbf{x}(t)$, $\mathbf{u}_i \sim \mathbf{u}(t)$, $t \in (t_i, t_{i+1}]$ and $\mathbf{x}_0 \sim \mathbf{x}(0)$

2. By denoting $h_i = t_{i+1} - t_i$, (**OCP**) is transcribed into the following nonlinear, constrained optimization problem

(NLOP)

$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i, t_i)$$

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h_i \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, t_i), \qquad i = 0, \dots, N-1$$

$$\mathbf{u}_i \in U, i = 0, \dots, N-1, \qquad F(\mathbf{x}_N) = 0$$

Transcription into nonlinear programming

Consistency of Time Discretization

Is this approximation consistent with the original formulation?

Yes!

Indeed, the KKT conditions for **(NLOP)** converge to the necessary optimality conditions for **(OCP)**, that are given by the Pontryagin's Minimum Principle, when $h_i \longrightarrow 0$

Forward Euler time discretization

1. Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and, for every $i = 0, \dots, N-1$, define $\mathbf{x}_{i+1} \sim \mathbf{x}(t)$, $\mathbf{u}_i \sim \mathbf{u}(t)$, $t \in (t_i, t_{i+1}]$ and $\mathbf{x}_0 \sim \mathbf{x}(0)$
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$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i, t_i)$$

(NLOP)

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h_i \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, t_i), \qquad i = 0, \dots, N-1$$

$$\mathbf{u}_i \in U \,, i = 0, \dots, N-1 \,, \qquad F(\mathbf{x}_N) = 0$$

Consistency of time discretization

Simplified Formulation

$$\min \int_0^{t_f} g\big(\mathbf{x}(t), \mathbf{u}(t)\big)\; dt$$

(**OCP**)

$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big),\; t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

Pontryagin's Minimum Principle (PMP)

Recall that the necessary optimality conditions for (OCP) are given by the following expressions

- Co-state equation:

$$\dot{\mathbf{p}}(t) = -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big(\mathbf{x}(t), \mathbf{u}(t)\big)' \mathbf{p}(t) - \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{u}(t))$$

- Control equation:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\big(\mathbf{x}(t), \mathbf{u}(t)\big)' \mathbf{p}(t) + \frac{\partial g}{\partial \mathbf{u}}\big(\mathbf{x}(t), \mathbf{u}(t)\big) = \mathbf{0}$$

Consistency of time discretization

Simplified Formulation

$$\min \int_0^{t_f} g\big(\mathbf{x}(t), \mathbf{u}(t)\big)\ dt$$

(OCP)

$$\dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t), \mathbf{u}(t)\big),\ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

Related non-linear program (NLOP)

After discretization in time:

$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i) \qquad \textbf{(NLOP)}$$

$$\mathbf{x}_i + h_i \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i) - \mathbf{x}_{i+1} = \mathbf{0}, \qquad i = 0, \dots, N-1$$

Consistency of time discretization

KKT Related to (NLOP)

Denote the Lagrangian related to **(NLOP)** as

$$\mathcal{L} = \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i) + \sum_{i=0}^{N-1} \boldsymbol{\lambda}_i'(\mathbf{x}_i + h_i \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i) - \mathbf{x}_{i+1})$$

Then, the KKT conditions related to **(NLOP)** read as:

- Derivative w.r.t. $\mathbf{x}_i$:

$$h_i \frac{\partial g}{\partial \mathbf{x}_i}(\mathbf{x}_i, \mathbf{u}_i) + \boldsymbol{\lambda}_i - \boldsymbol{\lambda}_{i-1} + h_i \frac{\partial \mathbf{f}}{\partial \mathbf{x}_i}(\mathbf{x}_i, \mathbf{u}_i)' \boldsymbol{\lambda}_i = \mathbf{0}$$

- Derivative w.r.t. $\mathbf{u}_i$:

$$h_i \frac{\partial g}{\partial \mathbf{u}_i}(\mathbf{x}_i, \mathbf{u}_i) + h_i \frac{\partial \mathbf{f}}{\partial \mathbf{u}_i}(\mathbf{x}_i, \mathbf{u}_i)' \boldsymbol{\lambda}_i = \mathbf{0}$$

Related non-linear program (NLOP)

After discretization in time:

$$\min_{(\mathbf{x}_i, \mathbf{u}_i)} \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i) \qquad \textbf{(NLOP)}$$

$$\mathbf{x}_i + h_i \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i) - \mathbf{x}_{i+1} = \mathbf{0}, \qquad i = 0, \ldots, N-1$$

Consistency of time discretization

KKT Related to (NLOP)

Denote the Lagrangian related to **(NLOP)** as

$$\mathcal{L} = \sum_{i=0}^{N-1} h_i g(\mathbf{x}_i, \mathbf{u}_i) + \sum_{i=0}^{N-1} \boldsymbol{\lambda}_i'(\mathbf{x}_i + h_i \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i) - \mathbf{x}_{i+1})$$

Then, the KKT conditions related to **(NLOP)** read as:

- Derivative w.r.t. $\mathbf{x}_i$:

$$h_i \frac{\partial g}{\partial \mathbf{x}_i}(\mathbf{x}_i, \mathbf{u}_i) + \boldsymbol{\lambda}_i - \boldsymbol{\lambda}_{i-1} + h_i \frac{\partial \mathbf{f}}{\partial \mathbf{x}_i}(\mathbf{x}_i, \mathbf{u}_i)' \boldsymbol{\lambda}_i = \mathbf{0}$$

- Derivative w.r.t. $\mathbf{u}_i$:

$$h_i \frac{\partial g}{\partial \mathbf{u}_i}(\mathbf{x}_i, \mathbf{u}_i) + h_i \frac{\partial \mathbf{f}}{\partial \mathbf{u}_i}(\mathbf{x}_i, \mathbf{u}_i)' \boldsymbol{\lambda}_i = \mathbf{0}$$

Consistency with the PMP

We finally obtain:

$$\frac{\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_{i-1}}{h_i} = -\frac{\partial \mathbf{f}}{\partial \mathbf{x}_i}(\mathbf{x}_i, \mathbf{u}_i)' \boldsymbol{\lambda}_i - \frac{\partial g}{\partial \mathbf{x}_i}(\mathbf{x}_i, \mathbf{u}_i)$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}_i}(\mathbf{x}_i, \mathbf{u}_i)' \boldsymbol{\lambda}_i + \frac{\partial g}{\partial \mathbf{u}_i}(\mathbf{x}_i, \mathbf{u}_i) = \mathbf{0}$$

Let $\mathbf{p}(t) = \boldsymbol{\lambda}_i$ for $t \in [t_i, t_{i+1}]$, $i = 0, \dots, N-1$ and $\mathbf{p}(0) = \boldsymbol{\lambda}_0$. Then, the equations above are the discretized version of the necessary conditions for **(OCP)**:

$$\dot{\mathbf{p}}(t) = -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big(\mathbf{x}(t), \mathbf{u}(t)\big)' \mathbf{p}(t) - \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{u}(t))$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\big(\mathbf{x}(t), \mathbf{u}(t)\big)' \mathbf{p}(t) + \frac{\partial g}{\partial \mathbf{u}}\big(\mathbf{x}(t), \mathbf{u}(t)\big) = \mathbf{0}$$

Solution approaches:

1. state and control parameterization methods

2. control parameterization methods

Example: Zermelo's Problem

- Designing direct methods in Matlab: transcribe optimal control problem into a non-linear program, and solve it via fmincon

Modified Zermelo's Problem

(OCP)

$$\min \int_0^{t_f} u(t)^2 \, dt$$

$$\dot{x}(t) = v \cos\big(u(t)\big) + \text{flow}\big(y(t)\big), t \in [0, t_f]$$

$$\dot{y}(t) = v \sin\big(u(t)\big), \; t \in [0, t_f]$$

$$(x, y)(0) = 0, \; (x, y)(t_f) = (M, \ell)$$

$$|u(t)| \leq 1, \; t \in [0, t_f]$$

State and control parameterization method

(NLOP)

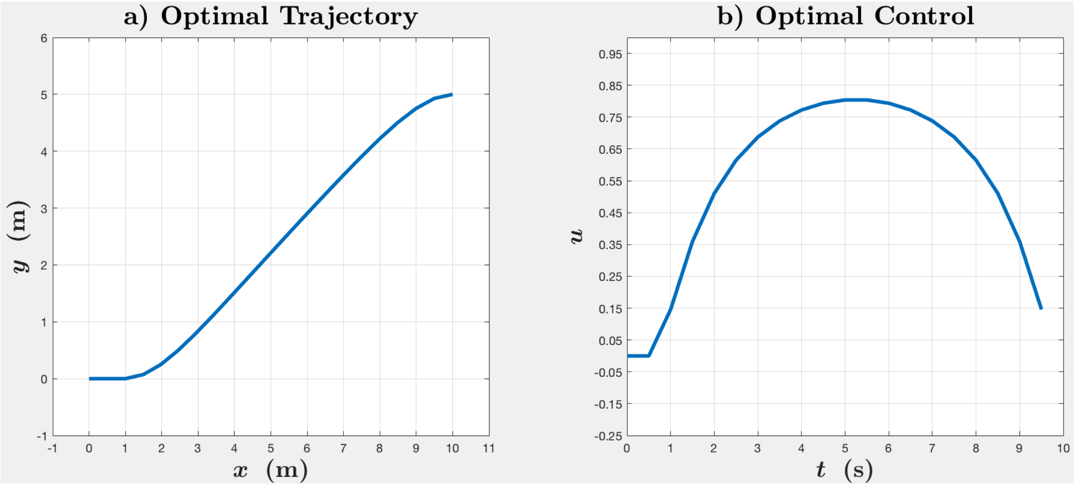
$$\min_{(x_i, u_i)} \sum_{i=0}^{N-1} u_i^2$$

$$x_{i+1} = x_i + h\big(v \cos(u_i) + \text{flow}(y_i)\big)$$

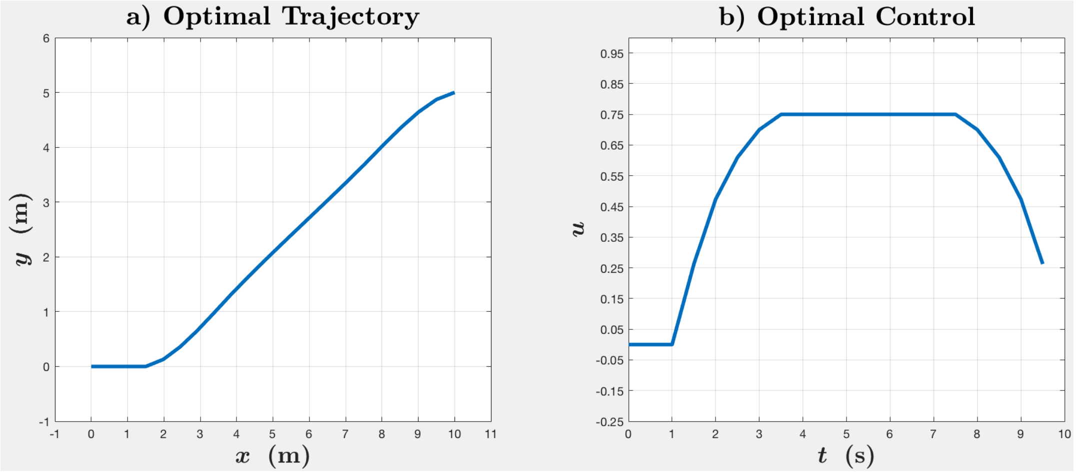
$$y_{i+1} = y_i + h \, v \sin(u_i) \, , |u_i| \leq u_{max}$$

$$(x_0, y_0) = 0 \, , (x_N, y_N) = (M, \ell)$$

Results



$|u(t)| \leq 1$
N = 20
28 iterations



$|u(t)| \leq 0.75$
N = 20
23 iterations

Transcription into nonlinear programming
(control parametrization method)

(OCP)

$$\min \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t),\ t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0,\ \ \mathbf{x}(t_f) \in M_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m,\ \ t \in [0, t_f]$$

Time and control discretization

1. Select a discretization $0 = t_0 < t_1 < \cdots < t_N = t_f$ for the interval $[0, t_f]$ and, for every $i = 0, \dots, N-1$, define $\mathbf{u}_i \sim \mathbf{u}(t)\ ,\ t \in (t_i, t_{i+1}]$
2. By denoting $h_i = t_{i+1} - t_i$, (**OCP**) is transcribed into the following nonlinear, constrained optimization problem

(NLOP-C)

$$\min_{\mathbf{u}_i} \sum_{i=0}^{N-1} h_i g(\mathbf{x}(t_i), \mathbf{u}_i, t_i)$$

$$\mathbf{u}_i \in U\ , i = 0, \dots, N-1\ , \qquad F(\mathbf{x}(t_N)) = 0$$

where each $\mathbf{x}(t_i)$ is recursively computed via

$$\mathbf{x}(t_{i+1}) = \mathbf{x}(t_i) + h_i \mathbf{f}(\mathbf{x}(t_i), \mathbf{u}_i, t_i), i = 0, \dots, N-1$$

Example: Zermelo's Problem

Modified Zermelo's Problem

Control parameterization method

(OCP)

$$\min \int_0^{t_f} u(t)^2 \, dt$$

$$\dot{x}(t) = v\cos\big(u(t)\big) + \text{flow}\big(y(t)\big), \, t \in [0, t_f]$$

$$\dot{y}(t) = v\sin\big(u(t)\big), \; t \in [0, t_f]$$

$$(x, y)(0) = 0\,, \; (x, y)(t_f) = (M, \ell)$$

$$|u(t)| \leq 1, \; t \in [0, t_f]$$

(NLOP-C)

$$\min_{u_i} \sum_{i=0}^{N-1} u_i^2$$

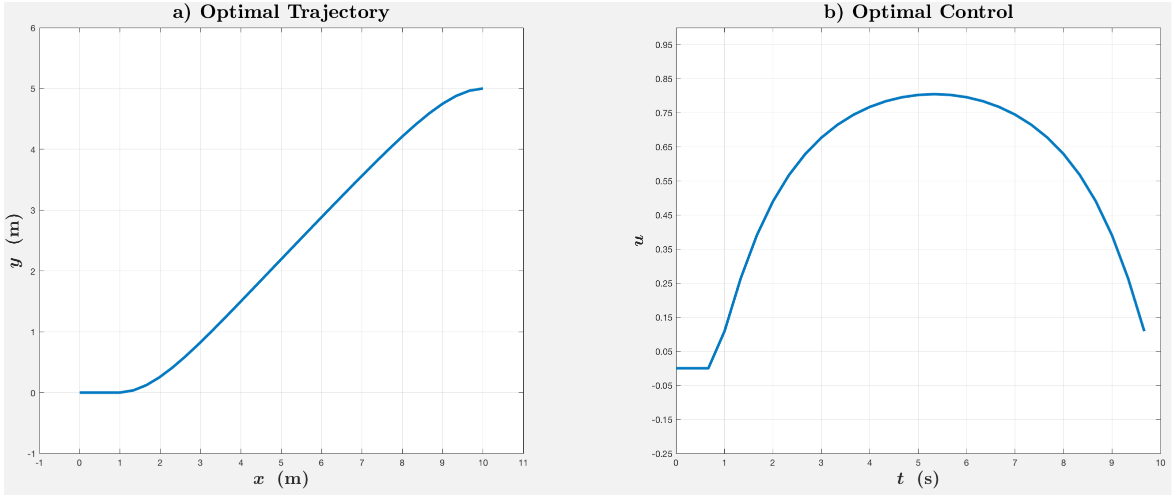
$$(x, y)(t_N) = (M, \ell), \quad |u_i| \leq u_{max}$$

where, recursively:

$$x(t_N) = x_0 + h_i \sum_{i=0}^{N-1} \big(v\cos(u_i) + \text{flow}(y(t_i))\big)$$

$$y(t_i) = y_0 + h_i \sum_{j=0}^{i} v\sin(u_j)$$

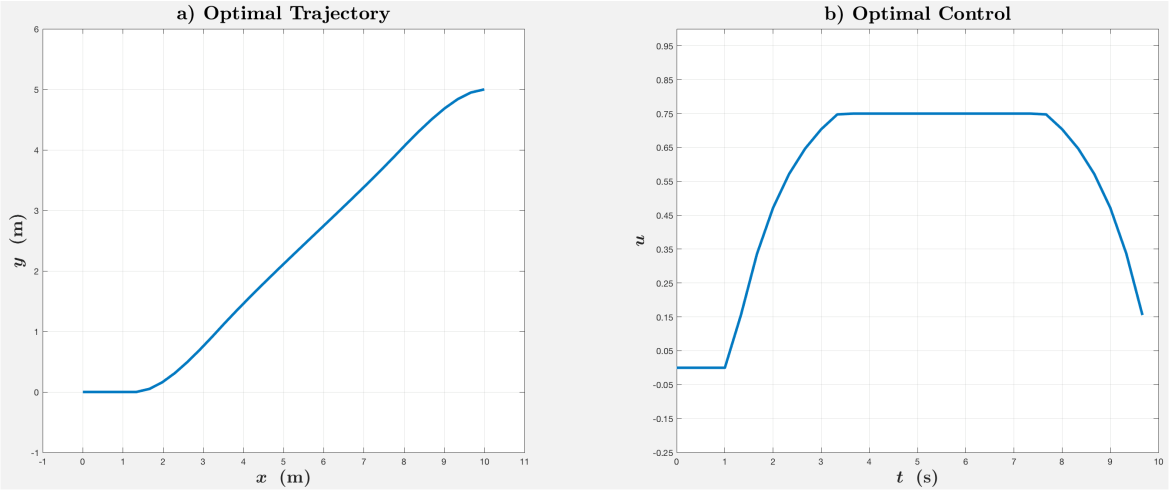
Results



$|u(t)| \leq 1$

N = 30

50 iterations



$|u(t)| \leq 0.75$

N = 30

16 iterations

Next time

- Direct collocation and SCP