AA203
Optimal and Learning-based Control

Numerical indirect methods for optimal control*

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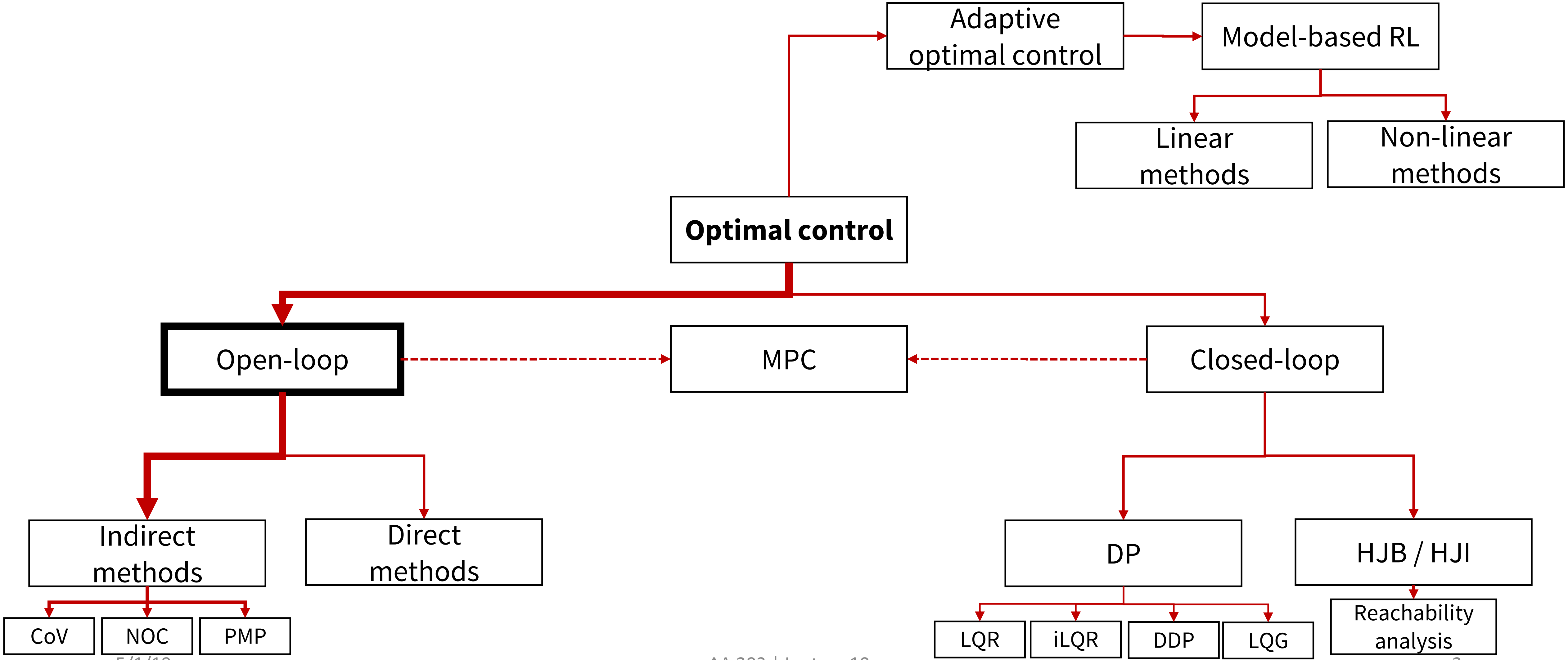
Autonomous Systems Lab

Roadmap


Adaptive optimal control
Model-based RL
Linear methods
Non-linear methods
Optimal control
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI
CoV
NOC
PMP
LQR
iLQR
DDP
LQG
Reachability analysis

Optimal Control Problem

(**OCP**)

$$\min \; h\left(\mathbf{x}(t_f)\right) + \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) \; dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad , \; t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0 \quad , \quad \mathbf{x}(t_f) \in M_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m \quad , \quad t \in [0, t_f]$$

For simplicity:

- h does not explicitly depend on t
- We assume $t_0 = 0$

- Indirect Methods:
  1. Apply necessary conditions for optimality to (**OCP**)
  2. Solve a two-point boundary value problem

- Direct Methods:
  1. Transcribe (**OCP**) into a nonlinear, constrained optimization problem
  2. Solve the optimization problem via nonlinear programming

1) Free Final Time and Fixed Final Point

(OCP)

$$\min \; h\left(\mathbf{x}(t_f)\right) + \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\; dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad , \; t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0 \quad , \quad \mathbf{x}(t_f) = \mathbf{x}_f$$

$$\mathbf{u}(t) \in U \subseteq \mathbb{R}^m \quad , \quad t \in [0, t_f]$$

Pontryagin's Minimum Principle

1. Define the Hamiltonian:
   $$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = \mathbf{p}'\mathbf{f}(\mathbf{x}, \mathbf{u}, t) + g(\mathbf{x}, \mathbf{u}, t)$$
2. Find candidates for optimal control solutions:
   $$\mathbf{u}^*(\mathbf{x}, \mathbf{p}, t) = \text{argmin}_{\mathbf{u} \in U} \; H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t)$$
3. Define the system of differential equations:
   $$\begin{cases} \dot{\mathbf{x}}(t) = \dfrac{\partial H}{\partial \mathbf{p}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \\ \dot{\mathbf{p}}(t) = -\dfrac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \end{cases}$$
   with conditions: $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{p}(0) = \mathbf{p}_0, \mathbf{x}(t_f) = \mathbf{x}_f$ and
   $$H\left(\mathbf{x}(t_f), \mathbf{p}(t_f), \mathbf{u}(t_f), t_f\right) = 0$$

1) Free Final Time and Fixed Final Point

Two-Point Boundary Value Problem

- Define the new variable and dynamics:

$$\mathbf{z} = (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n},$$

$$\mathbf{R}(\mathbf{z}, t) = \left(\frac{\partial H}{\partial \mathbf{p}}(\mathbf{z}, \mathbf{u}^*(\mathbf{z}, t), t), -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{z}, \mathbf{u}^*(\mathbf{z}, t), t) \right)$$

- For every value $\mathbf{p}_0 \in \mathbb{R}^n$ and final time t_f, denote $\mathbf{z}_{\mathbf{p}_0}^{t_f}(\cdot)$ the solution of:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{R}(\mathbf{z}(t), t), \quad t \in [0, t_f] \\ \mathbf{z}(0) = (\mathbf{x}_0, \mathbf{p}_0) \end{cases}$$

- Denote $proj_{\mathbf{x}}(\mathbf{x}, \mathbf{p}) = \mathbf{x}$ and define the function:

$$S : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$$
$$(\mathbf{p}_0, t_f) \longmapsto \left(proj_{\mathbf{x}}\left(\mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f)\right) - \mathbf{x}_f, H(t_f) \right)$$

Pontryagin's Minimum Principle

1. Define the Hamiltonian:

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = \mathbf{p}'\mathbf{f}(\mathbf{x}, \mathbf{u}, t) + g(\mathbf{x}, \mathbf{u}, t)$$

2. Find candidates for optimal control solutions:

$$\mathbf{u}^*(\mathbf{x}, \mathbf{p}, t) = \text{argmin}_{\mathbf{u} \in U} \; H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t)$$

3. Define the system of differential equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \\ \dot{\mathbf{p}}(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \end{cases}$$

with conditions: $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{p}(0) = \mathbf{p}_0, \mathbf{x}(t_f) = \mathbf{x}_f$ and $H(\mathbf{x}(t_f), \mathbf{p}(t_f), \mathbf{u}(t_f), t_f) = 0$

Find the value of $\mathbf{p}_0$ that satisfies the system above

1) Free Final Time and Fixed Final Point

Two-Point Boundary Value Problem

- Define the new variable and dynamics:

$$\mathbf{z} = (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n},$$

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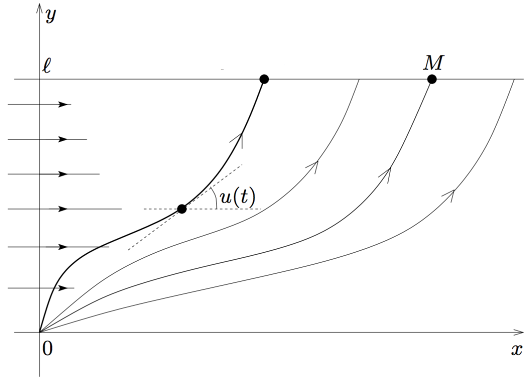
Example: Zermelo's Problem

$$\min \int_0^{t_f} 1 \, dt$$

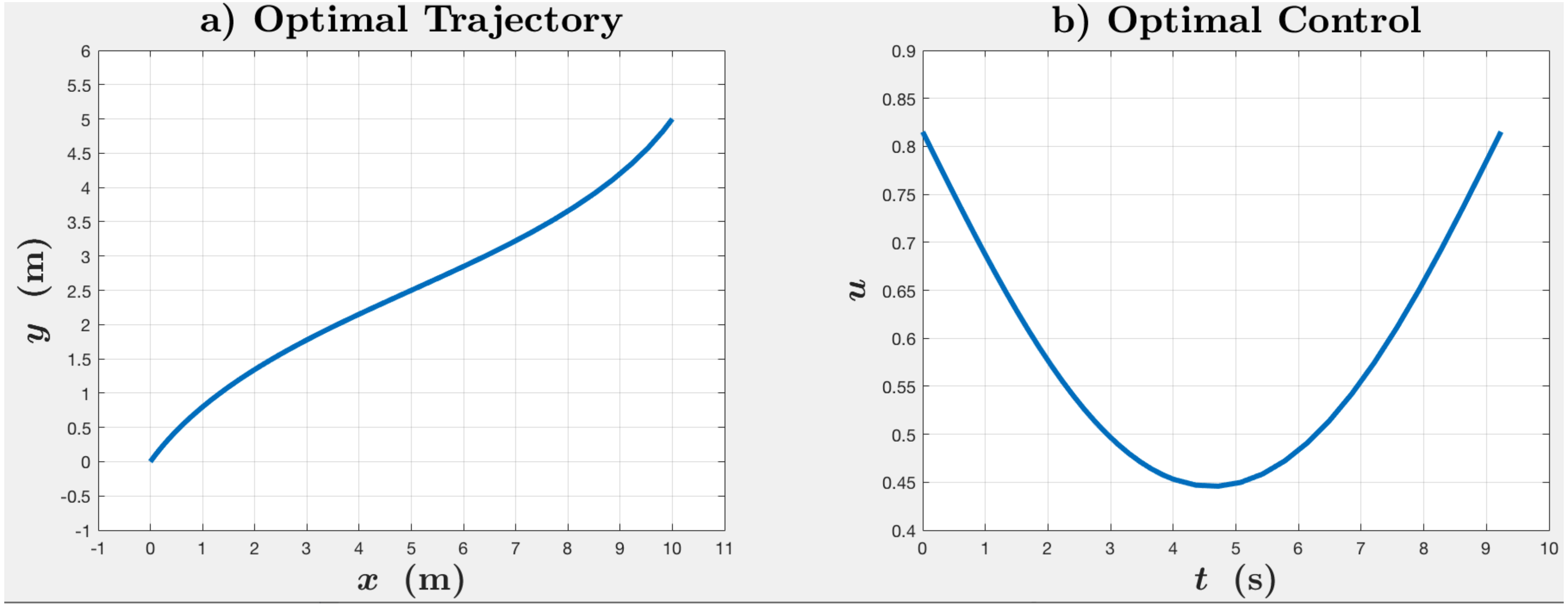
$$\dot{x}(t) = v\cos(u(t)) + fl(y(t)), \; t \in [0, t_f]$$
$$\dot{y}(t) = v\sin(u(t)), \quad t \in [0, t_f]$$
$$(x, y)(0) = 0, \; (x, y)(t_f) = (M, \ell)$$
$$u(t) \in \mathbb{R}, \; t \in [0, t_f]$$



Recall that:

- $H(x, y, p_x, p_y, u) = (p_x, p_y) \cdot f(x, y, u) + 1$
- $u^*(x, y, p_x, p_y) = \text{argmin}_{u \in \mathbb{R}} H(x, y, p_x, p_y, u)$

1) Free Final Time and Fixed Final Point



2) Free Final Time and Free Final Point

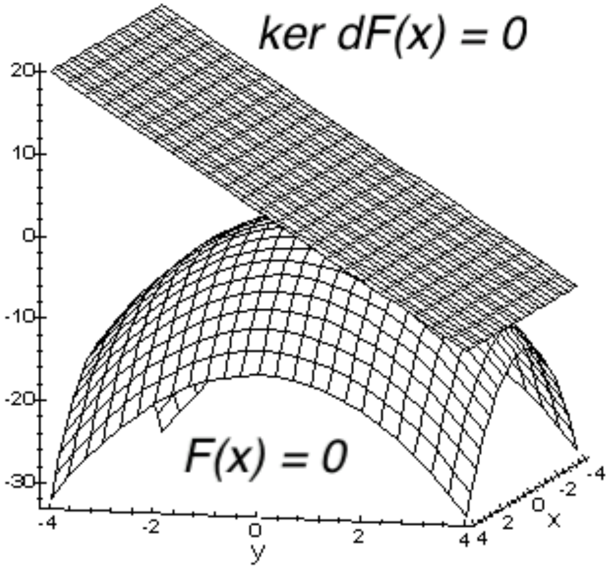
(OCP)

$$\min\ h\left(\mathbf{x}(t_f)\right) + \int_0^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\ dt$$

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad t \in [0, t_f]$$

$$\mathbf{x}(0) = \mathbf{x}_0, \mathbf{u}(t) \in U \subseteq \mathbb{R}^m, \quad t \in [0, t_f]$$

$$\mathbf{x}(t_f) \in M_f := \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 0\}$$



Pontryagin's Minimum Principle

1. Define the Hamiltonian:
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2. Find candidates for optimal control solutions:
$$\mathbf{u}^*(\mathbf{x}, \mathbf{p}, t) = \text{argmin}_{u \in U}\ H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t)$$

3. Define the system of differential equations:
$$\begin{cases} \dot{\mathbf{x}}(t) = \dfrac{\partial H}{\partial \mathbf{p}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \\ \dot{\mathbf{p}}(t) = -\dfrac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}^*(\mathbf{x}(t), \mathbf{p}(t), t), t) \end{cases}$$

with conditions: $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{p}(0) = \mathbf{p}_0, \mathbf{x}(t_f) = \mathbf{x}_f$, $H\left(\mathbf{x}(t_f), \mathbf{p}(t_f), \mathbf{u}(t_f), t_f\right) = 0$ and:

$$\mathbf{p}(t_f) - \nabla h(\mathbf{x}(t_f)) \perp \ker dF(\mathbf{x}(t_f))$$

2) Free Final Time and Free Final Point

Two-Point Boundary Value Problem

- Define the new variable and dynamics:

$$\mathbf{z} = (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n},$$

$$\mathbf{R}(\mathbf{z}, t) = \left(\frac{\partial H}{\partial \mathbf{p}}(\mathbf{z}, \mathbf{u}^*(\mathbf{z}, t), t), -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{z}, \mathbf{u}^*(\mathbf{z}, t), t) \right)$$

- For every value $\mathbf{p}_0 \in \mathbb{R}^n$ and final time t_f, denote $\mathbf{z}_{\mathbf{p}_0}^{t_f}(\cdot)$ the solution of:

$$\begin{cases} \dot{\mathbf{z}}(t) = \mathbf{R}(\mathbf{z}(t), t), \quad t \in [0, t_f] \\ \mathbf{z}(0) = (\mathbf{x}_0, \mathbf{p}_0) \end{cases}$$

- Denote $proj_{\mathbf{p}}(\mathbf{x}, \mathbf{p}) = \mathbf{p}$ and $\mathbf{x}(t_f) = proj_{\mathbf{x}}\left(\mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f)\right)$.

  Define the function:

$$S : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$$
$$(\mathbf{p}_0, t_f) \mapsto \left(\left(proj_{\mathbf{p}}\left(\mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f)\right) - \nabla h\left(\mathbf{x}(t_f)\right) \right) \cdot \ker dF\left(\mathbf{x}(t_f)\right), F\left(\mathbf{x}(t_f)\right), H(t_f) \right)$$

Pontryagin's Minimum Principle

1. Define the Hamiltonian:

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t) = \mathbf{p}'\mathbf{f}(\mathbf{x}, \mathbf{u}, t) + g(\mathbf{x}, \mathbf{u}, t)$$

2. Find candidates for optimal control solutions:

$$\mathbf{u}^*(\mathbf{x}, \mathbf{p}, t) = \operatorname{argmin}_{\mathbf{u} \in U} H(\mathbf{x}, \mathbf{p}, \mathbf{u}, t)$$

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with conditions: $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{p}(0) = \mathbf{p}_0, \mathbf{x}(t_f) = \mathbf{x}_f$, $H\left(\mathbf{x}(t_f), \mathbf{p}(t_f), \mathbf{u}(t_f), t_f\right) = 0$ and:

$$\mathbf{p}(t_f) - \nabla h(\mathbf{x}(t_f)) \perp \ker dF(\mathbf{x}(t_f))$$

Find the value of p_0 that satisfies the system above

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- For every value $\mathbf{p}_0 \in \mathbb{R}^n$ and final time t_f, denote $\mathbf{z}_{\mathbf{p}_0}^{t_f}(\cdot)$ the solution of:

$$\begin{cases} \dot{\mathbf{z}}(t) = R(\mathbf{z}(t), t), \quad t \in [0, t_f] \\ \mathbf{z}(0) = (\mathbf{x}_0, \mathbf{p}_0) \end{cases}$$

- Denote $proj_{\mathbf{p}}(\mathbf{x}, \mathbf{p}) = \mathbf{p}$ and $\mathbf{x}(t_f) = proj_{\mathbf{x}}\left(\mathbf{z}_{\mathbf{p}_0}^{t_f}(t_f)\right)$.

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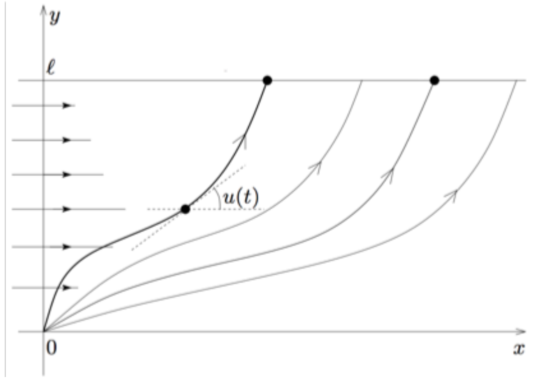
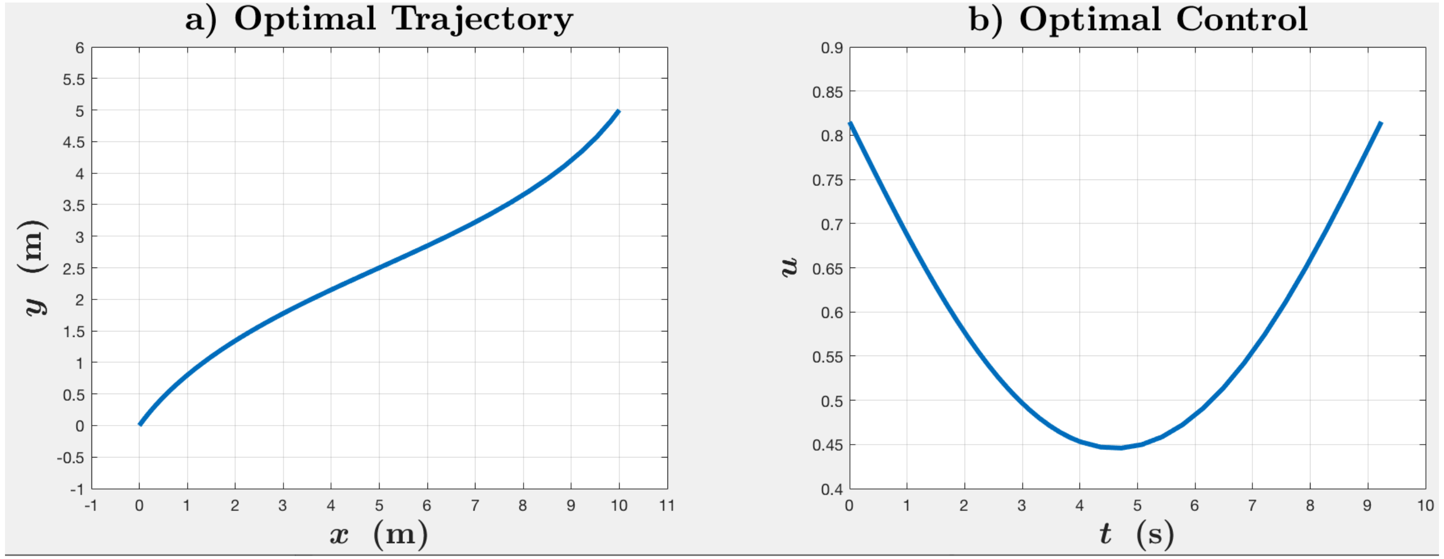
$$\dot{y}(t) = v\sin(u(t)), \quad t \in [0, t_f]$$
$$(x, y)(0) = 0, \; u(t) \in \mathbb{R}, \quad t \in [0, t_f]$$
$$(x, y)(t_f) \in M_f = \{ (x, y) \in \mathbb{R}^2 : F(x, y) = y - \ell = 0 \}$$



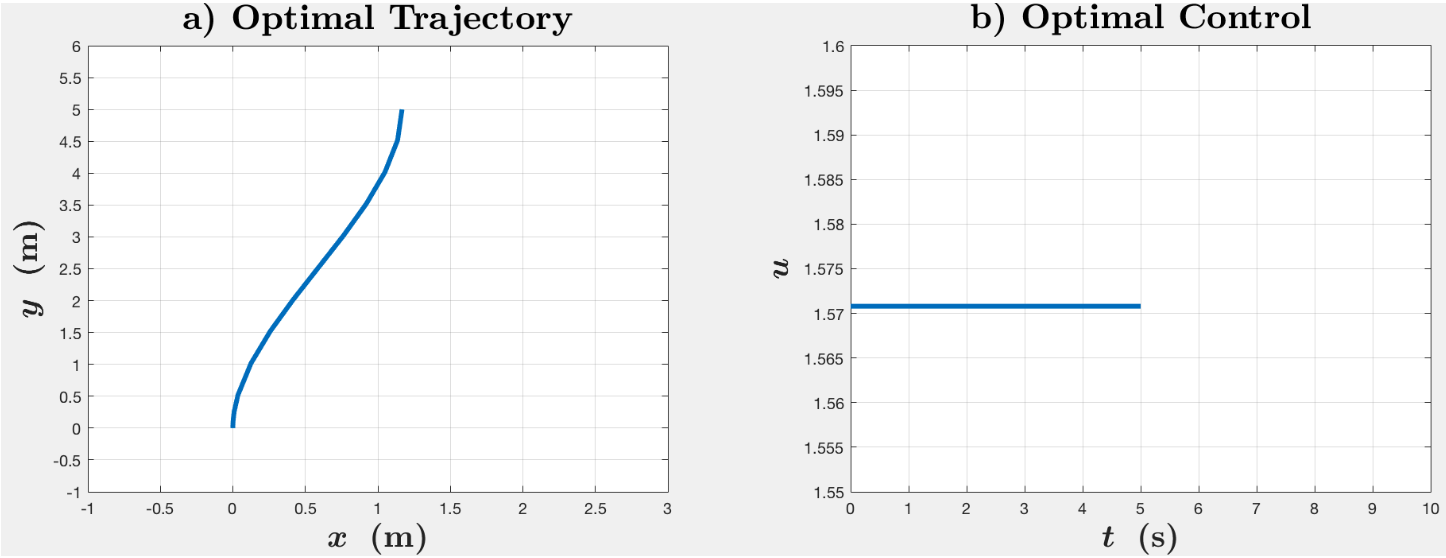
Recall that:

- $H(x, y, p_x, p_y, u) = (p_x, p_y) \cdot f(x, y, u) + 1$
- $u(x, y, p_x, p_y) = \text{argmin}_{u \in \mathbb{R}} H(x, y, p_x, p_y, u)$

2) Free Final Time and Free Final Point



Fixed Final Point



Free Final Point

Next time

- Introduction to direct methods for optimal control