# Principles of Robot Autonomy I

Trajectory tracking

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# Agenda

- Trajectory tracking
  - Based on differential flatness techniques
  - Based on LQR techniques

- Readings
  - Chapter 3 in PoRA lecture notes

# Trajectory tracking

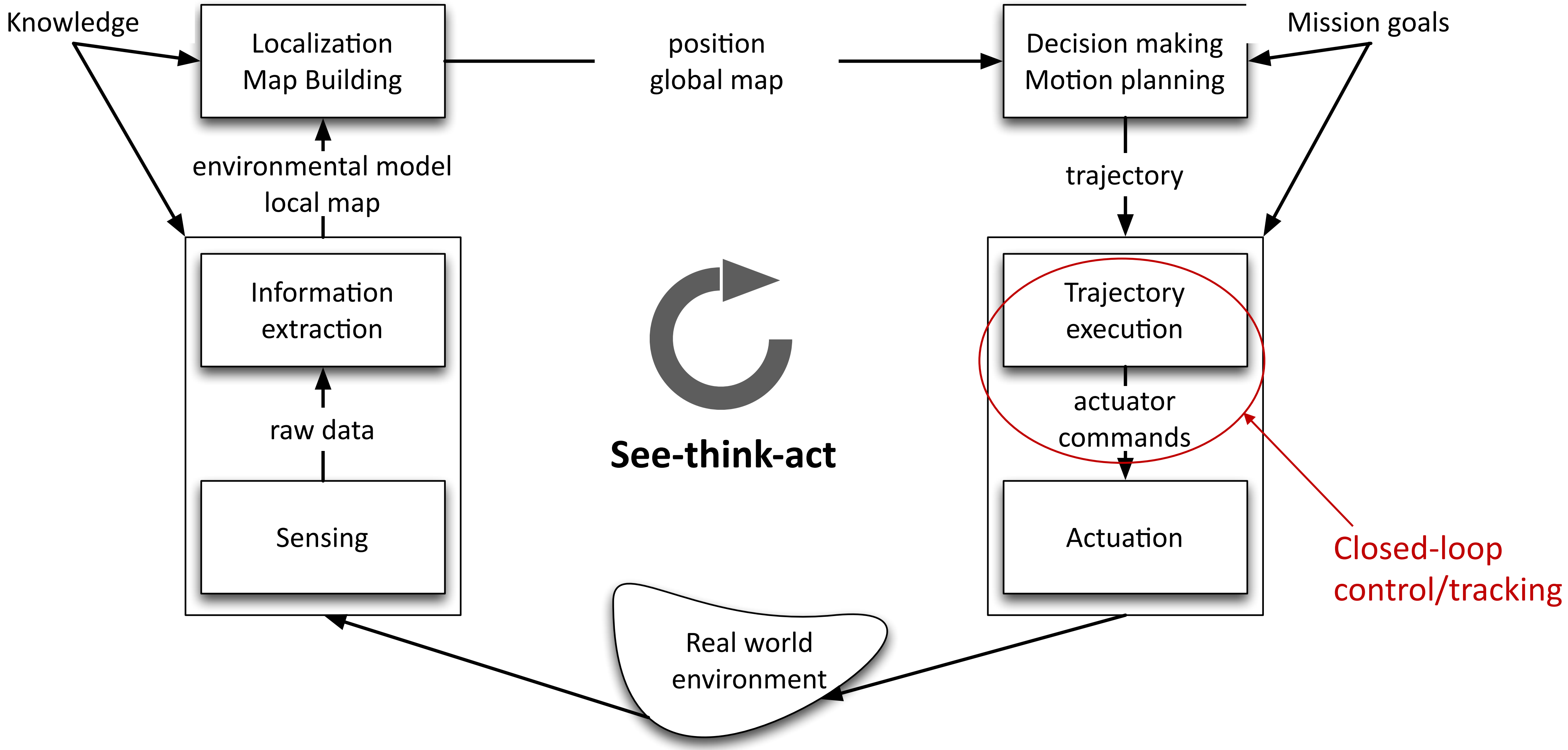
- Back to two-step design strategy

Tracking control law

$$\mathbf{u}^*(t) = \mathbf{u}_d(t) + \boxed{\pi(\mathbf{x}(t), \mathbf{x}(t) - \mathbf{x}_d(t))}$$

- Reference trajectory and control history (i.e., $\mathbf{x}_d(t)$ and $\mathbf{u}_d(t)$) are computed via open-loop techniques (e.g., differential flatness)
- For reference tracking (Problem 3 in pset 1)
  - Geometric (e.g., pursuit) strategies
  - Linearization (either approximate or exact) + linear structure
  - Non-linear control
  - Optimization-based techniques (e.g., MPC)

# The see-think-act cycle



# Differential flatness (recap)

- A nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ is differentially flat if there exists a set of outputs $\mathbf{z} = \alpha(\mathbf{x}, \mathbf{u}, \dots, \mathbf{u}^{(p)})$ such that

$$\mathbf{x} = \beta(\mathbf{z}, \dot{\mathbf{z}}, \dots, \mathbf{z}^{(q)})$$

$$\mathbf{u} = \gamma(\mathbf{z}, \dot{\mathbf{z}}, \dots, \mathbf{z}^{(q)})$$

- One can then use any interpolation scheme (e.g., piecewise polynomial (spline)) to plan the trajectory of $\mathbf{z}$ in such a way as to satisfy the appropriate boundary conditions
- The evolution of the state variables $\mathbf{x}$, together with the associated control inputs $\mathbf{u}$, can then be computed algebraically from $\mathbf{z}$

# Trajectory tracking for differentially flat systems

- Example: dynamically extended unicycle model

$$\dot{x}(t) = V\cos(\theta(t))$$
$$\dot{y}(t) = V\sin(\theta(t))$$
$$\dot{V}(t) = a(t)$$
$$\dot{\theta}(t) = \omega(t)$$

- The system is differentially flat with flat outputs $(x, y)$, in particular

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -V\sin(\theta) \\ \sin(\theta) & V\cos(\theta) \end{bmatrix}}_{:=J} \begin{bmatrix} a \\ \omega \end{bmatrix} := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

# Trajectory tracking for differentially flat systems

- Then one can use the following virtual control law for trajectory tracking:

$$w_1 = \ddot{x}_d + k_{px}(x_d - x) + k_{dx}(\dot{x}_d - \dot{x})$$
$$w_2 = \ddot{y}_d + k_{py}(y_d - y) + k_{dy}(\dot{y}_d - \dot{y})$$

where $k_{px}, k_{dx}, k_{py}, k_{dy} > 0$ are control gains

- Such a law guarantees exponential convergence to zero of the Cartesian tracking error

# Trajectory tracking for differentially flat systems

- More broadly, suppose system is differentially flat: the full state and control trajectories can be computed from flat outputs $(\mathbf{z}, \dot{\mathbf{z}}, \ldots, \mathbf{z}^{(q)})$
- Define: $\mathbf{z}^{(q+1)} = \mathbf{w}$
- One can then design a tracking controller by using linear control techniques; in particular, for a given reference flat output $\mathbf{z}_d$, define the *component-wise* error

$$e_i := z_i - z_{i,d}, \text{ which implies } e_i^{(q+1)} = w_i - w_{i,d}$$

- For guaranteed convergence to zero of tracking error, one can set

$$w_i = w_{i,d} - \sum_{j=0}^{q} k_{i,j} e_i^{(j)},$$

with the gains $\{k_{i,j}\}$ chosen so as to enforce stability

# LQR-based methods

- The previous approach only works for differentially flat systems

- How can we control more general classes of systems?
  - Nonlinear control techniques
  - Linear-quadratic regulation (LQR) methods

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# Linear-quadratic regulator (LQR)

- How can we regulate (i.e., drive to the origin) the linear system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ with minimum control effort?
- We define the optimal control problem

$$\begin{aligned} \min_{\mathbf{u}} \quad & \mathbf{x}(T)'F\mathbf{x}(T) + \int_0^T (\mathbf{x}(t)'Q\mathbf{x}(t) + \mathbf{u}(t)'R\mathbf{u}(t))dt \\ \text{s.t.} \quad & \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \end{aligned}$$

- The optimal solution is of the form $\mathbf{u}_t = K_t\mathbf{x}_t$ where $K_t = -R^{-1}B'P_t$ and the matrix $P_t$ solves the continuous time Riccati diff. equation:

$$\dot{P}_t = -A'P_t - P_tA + P_tBR^{-1}B'P_t - Q \text{ with } P_T = F$$

- Note: this results holds even in the more general case $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$ (just plug $A(t)$ and $B(t)$ in the Riccati equation)

# Tracking LQR – linear case

- Consider the linear system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ and assume we would like to track a reference trajectory $\big(\mathbf{x}_d(t), \mathbf{u}_d(t)\big)_t$
- Define error variables $\delta\mathbf{x}(t) \coloneqq (\mathbf{x}(t) - \mathbf{x}_d(t))$ and $\delta\mathbf{u}(t) \coloneqq (\mathbf{u}(t) - \mathbf{u}_d(t))$, which leads to the dynamical system
$$\delta\dot{\mathbf{x}}(t) = A\delta\mathbf{x}(t) + B\delta\mathbf{u}(t)$$
- Define optimal control problem

$$\begin{aligned} \min_{\mathbf{u}} \quad & \delta\mathbf{x}(T)'F\,\delta\mathbf{x}(T) + \int_0^T (\delta\mathbf{x}(t)'Q\,\delta\mathbf{x}(t) + \delta\mathbf{u}(t)'R\,\delta\mathbf{u}(t))dt \\ \text{s.t.} \quad & \delta\dot{\mathbf{x}}(t) = A\delta\mathbf{x}(t) + B\delta\mathbf{u}(t) \end{aligned}$$

- Optimal solution is $\delta\mathbf{u}_t = K_t\,\delta\mathbf{x}_t$ (same $K_t$ as before), which leads to control $\mathbf{u}(t) = \mathbf{u}_d(t) + \delta\mathbf{u}(t)$

# Tracking LQR – nonlinear case

- Consider the non-linear system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ and assume we would like to track a reference trajectory $\left(\mathbf{x}_d(t), \mathbf{u}_d(t)\right)_t$
- Key idea: make the system "linear" by linearizing around $\left(\mathbf{x}_d(t), \mathbf{u}_d(t)\right)_t$:

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}\left(\mathbf{x}_d(t), \mathbf{u}_d(t)\right) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\left(\mathbf{x}_d(t), \mathbf{u}_d(t)\right)\left(\mathbf{x}(t) - \mathbf{x}_d(t)\right) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\left(\mathbf{x}_d(t), \mathbf{u}_d(t)\right)\left(\mathbf{u}(t) - \mathbf{u}_d(t)\right)$$

$$= \mathbf{x}_d(t) + A(t)\left(\mathbf{x}(t) - \mathbf{x}_d(t)\right) + B(t)\left(\mathbf{u}(t) - \mathbf{u}_d(t)\right)$$

- As before, we get a linear system in the error variables:

$$\delta\dot{\mathbf{x}}(t) = A(t)\delta\mathbf{x}(t) + B(t)\delta\mathbf{u}(t)$$

- Optimal solution is $\delta\mathbf{u}_t = K_t\,\delta\mathbf{x}_t$ which leads to control $\mathbf{u}(t) = \mathbf{u}_d(t) + \delta\mathbf{u}(t)$ (where $K_t$ is again obtained from the Riccati eq.)

# Transferring results to discrete case

- Same ideas apply for discrete-time systems, i.e., $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k)$
- Key difference: control is $\mathbf{u}_k = \mathbf{u}_k^d + \delta\mathbf{u}_k$, with $\delta\mathbf{u}_k = K_k\, \delta\mathbf{x}_k$, where $K_k = -(R + B_k' P_{k+1} B_k)^{-1} B_k' P_{k+1} A_k$ and $P_k$ is iteratively obtained from the discrete-time Riccati equation:

  $P_k = A_k' P_{k+1} A_k - A_k' P_{k+1} B_k (R + B_k' P_{k+1} B_k)^{-1} B_k' P_{k+1} A_k + Q$ with $P_N = F$
- Similarly as before:
  - $A_k = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_k^d, \mathbf{u}_k^d)$ and $B_k = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_k^d, \mathbf{u}_k^d)$
  - $R, Q, F$ have the same interpretation as before, namely tracking penalty, control effort penalty, and final tracking error penalty

# Next time