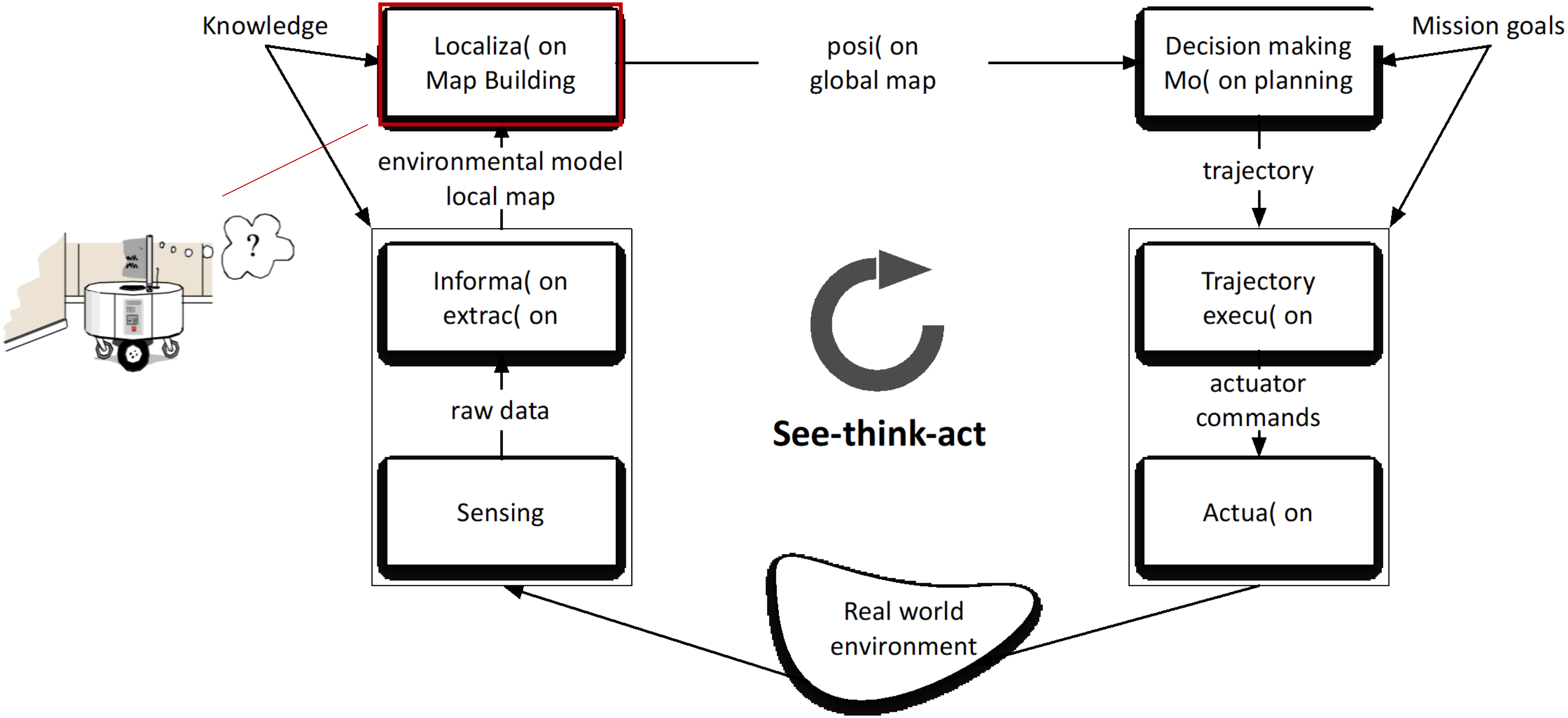
Principles of Robot Autonomy I

Introduction to state estimation and filtering theory

Stanford University

Autonomous Systems Lab

The see-think-act cycle



Knowledge

Localization
Map Building

position
global map

Decision making
Motion planning

Mission goals

environmental model
local map

trajectory

Information extraction

Trajectory execution

raw data

actuator
commands

Sensing

Actuation

See-think-act

Real world
environment

Agenda

- Agenda
  - Basic concepts about Bayesian filtering

- Readings:
  - Chapter 13 in PoRA lecture notes

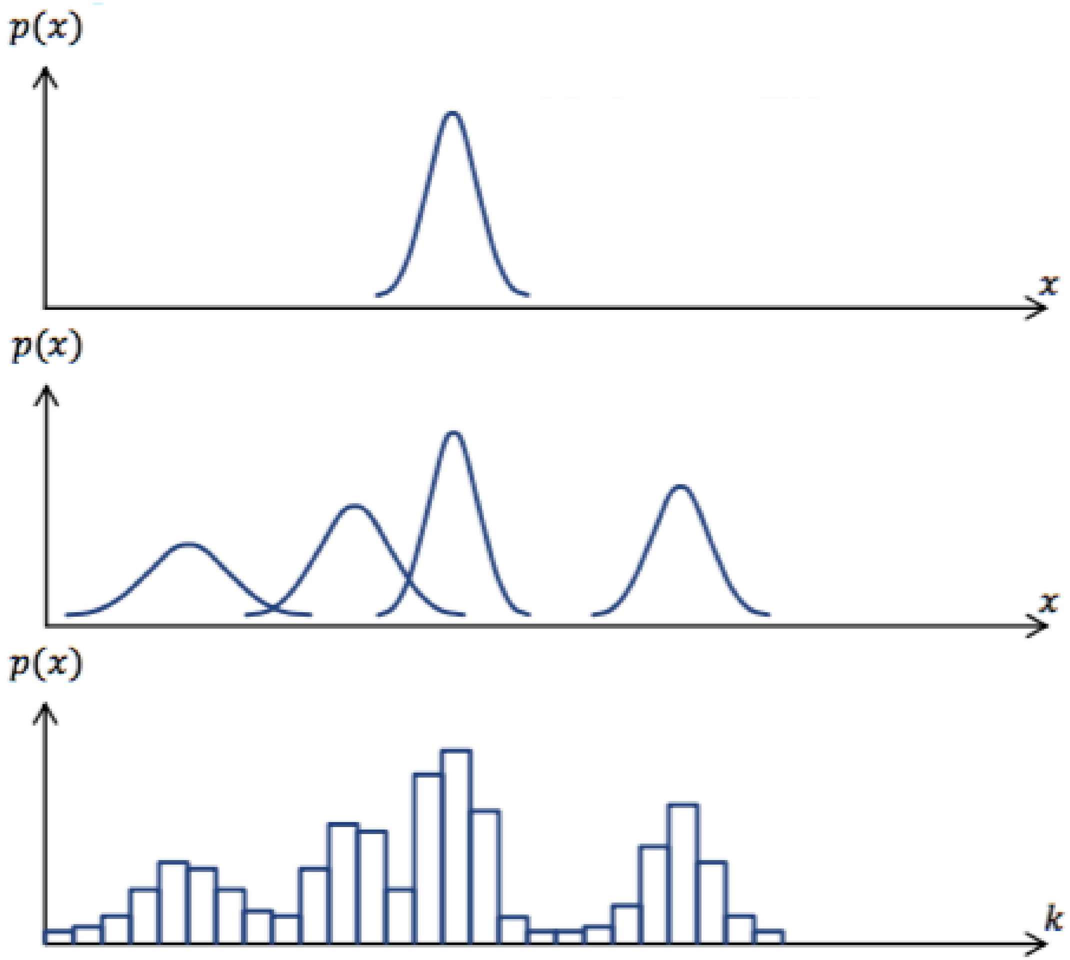
Map-based localization

- Key idea: robot *explicitly* attempts to localize by collecting sensor data, then updating belief about its position with respect to a map

- Two main aspects:
    - *Map representation*: how to represent the environment?
    - *Belief representation*: how to model the belief regarding the position within the map?

Probabilistic map-based localization

- Key idea: represent belief as a probability distribution
  1. Encodes sense of position
  2. Maintains notion of robot’s uncertainty
- Belief representation:
  1. Single-hypothesis vs. multiple hypothesis
  2. Continuous vs. discretized
- Today we will overview basic concepts in Bayesian filtering



Basic concepts in probability

- **Key idea**: quantities such as sensor measurements, states of a robot, and its environment are modeled as **random variables (RVs)**
- **Discrete RV**: the space of all the values that a random variable *X* can take on is *discrete*; characterized by probability mass function (pmf)

$$p(X = x) \quad (\text{or } p(x)), \qquad \sum_x p(X = x) = 1$$

Random variable ↗ ↖ Specific value

- **Continuous RV**: the space of all the values that a random variable *X* can take on is *continuous;* characterized by probability density function (pdf)

$$P(a \leq X \leq b) = \int_a^b p(x)\,dx, \qquad \int_{-\infty}^{\infty} p(x)\,dx = 1$$

Joint distribution, independence, and conditioning

- Joint distribution of two random variables *X* and *Y* is denoted as

$$p(x, y) := p(X = x \text{ and } Y = y)$$

- If *X* and *Y* are independent

$$p(x, y) = p(x)p(y)$$

- Suppose we know that $Y = y$ (with $p(y) > 0$); conditioned on this fact, the probability that the X's value is x is given by

$$p(x \,|\, y) := \frac{p(x, y)}{p(y)}$$

Conditional probability

Note: if *X* and *Y* are independent

$$p(x \,|\, y) := p(x)!$$

Law of total probability

- For discrete RVs:

$$p(x) = \sum_y p(x, y) = \sum_y p(x \mid y) p(y)$$

- For continuous RVs:

$$p(x) = \int p(x, y) dy = \int p(x \mid y) p(y) dy$$

- Note: if $p(y) = 0$, define the product $p(x \mid y)p(y) = 0$

Bayes' rule

- Key relation between $p(x \mid y)$ and its "inverse," $p(y \mid x)$
- For discrete RVs:

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} = \frac{p(y \mid x)p(x)}{\sum_{x'} p(y \mid x')p(x')}$$

- For continuous RVs:

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} = \frac{p(y \mid x)p(x)}{\int p(y \mid x')p(x')\, dx'}$$

Bayes’ rule and probabilistic inference

- Assume *x* is a quantity we would like to infer from *y*
- Bayes rule allows us to do so through the inverse probability, which specifies the probability of data *y* assuming that *x* was the cause

Posterior probability distribution

Prior probability distribution

$$p(x\,|\,y) = \frac{p(y\,|\,x)p(x)}{\int p(y\,|\,x')p(x')\,dx'}$$

Data

Normalizer, does not depend on $x := \eta^{-1}$

- Notational simplification

$$p(x\,|\,y) = \eta\, p(y\,|\,x)p(x)$$

More on Bayes' rule and independence

- Extension of Bayes rule: conditioning Bayes rule on *Z=z* gives

$$p(x \mid y, z) = \frac{p(y \mid x, z)p(x \mid z)}{p(y \mid z)}$$

- Extension of independence: *conditional independence*

$$p(x, y \mid z) = p(x \mid z)p(y \mid z), \qquad \text{equivalent to } \begin{cases} p(x \mid z) = p(x \mid z, y) \\ p(y \mid z) = p(y \mid z, x) \end{cases}$$

- Note: in general

$$p(x, y \mid z) = p(x \mid z)p(y \mid z) \;\not\Rightarrow\; p(x, y) = p(x)p(y)$$

$$p(x, y) = p(x)p(y) \;\not\Rightarrow\; p(x, y \mid z) = p(x \mid z)p(y \mid z)$$

Expectation of a RV

- Expectation for discrete RVs: $E[X] = \sum_x x\, p(x)$
- Expectation for continuous RVs: $E[X] = \int x\, p(x)\, dx$
- Expectation is a linear operator: $E[aX + b] = a\, E[X] + b$
- Expectation of a vector of RVs is simply the vector of expectations
- Covariance

$$\operatorname{cov}(X, Y) = E[(X - E[X])(Y - E[Y])^T] = E[XY^T] - E[X]E[Y]^T$$

Model for robot-environment interaction

- Two fundamental types of robot-environment interactions: the robot can influence the state of its environment through control actions, and gather information about the state through measurements
- State x_t: collection at time *t* of all aspects of the robot and its environment that can impact the future
    - Robot pose (e.g., robot location and orientation)
    - Robot velocity
    - Locations and features of surrounding objects in the environment, etc.
- Useful notation: $x_{t_1:t_2} := x_{t_1}, x_{t_1+1}, x_{t_1+2}, \ldots, x_{t_2}$
- A state x_t is called *complete* if no variables prior to x_t can influence the evolution of future states → Markov property

Measurement and control data

- Measurement data z_t: information about state of the environment at time *t*; useful notation

$$z_{t_1:t_2} := z_{t_1}, z_{t_1+1}, z_{t_1+2}, \ldots, z_{t_2}$$

- Control data u_t: information about the change of state at time *t*; useful notation

$$u_{t_1:t_2} := u_{t_1}, u_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2}$$

State equation

- General probabilistic generative model

$$p(x_t \,|\, x_{0:t-1},\, z_{1:t-1},\, u_{1:t})$$

Convention: first take control action and then take measurement

- Key assumption: state is complete (i.e., the Markov property holds)

State transition probability

$$p(x_t \,|\, x_{0:t-1},\, z_{1:t-1},\, u_{1:t}) = p(x_t \,|\, x_{t-1}, u_t)$$

- In other words, we assume *conditional independence*, with respect to conditioning on x_{t-1}

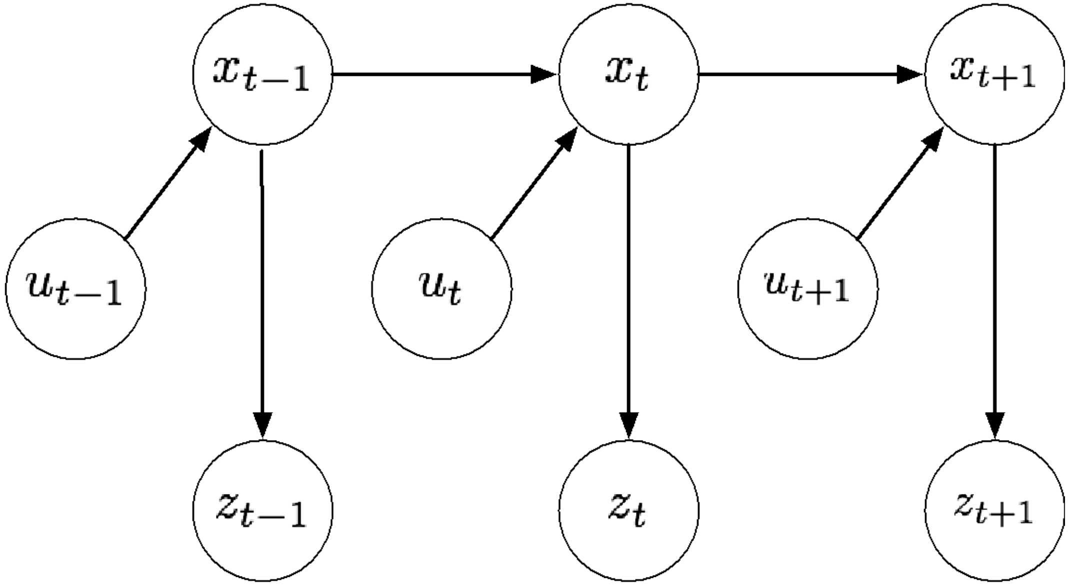
Measurement equation and overall stochastic model

- Assuming x_t is complete

$$p(z_t \mid x_{0:t},\, z_{1:t-1},\, u_{1:t}) = p(z_t \mid x_t)$$

Measurement probability

- Overall dynamic Bayes network model (also referred to as hidden Markov model)



Belief distribution

- Belief distribution: reflects internal knowledge about the state
- A belief distribution assigns a probability to each possible hypothesis with regard to the true state
- Formally, belief distributions are posterior probabilities over state variables conditioned on the available data

$$bel(x_t) := p(x_t \mid z_{1:t},\, u_{1:t})$$

- Similarly, the *prediction* distribution is defined as

$$\overline{bel}(x_t) := p(x_t \mid z_{1:t-1},\, u_{1:t})$$

- Calculating $bel(x_t)$ from $\overline{bel}\ (x_t)$ is called correction or measurement update

Bayes filter algorithm

- Bayes' filter algorithm: most general algorithm for calculating beliefs
- Key assumption: state is complete

- Recursive algorithm
  - Step 1 (prediction): compute $\overline{bel}\ (x_t)$
  - Step 2 (measurement update): compute $bel(x_t)$
- Algorithm initialized with $bel(x_0)$ (e.g., uniform or points mass)

Update rule

```
Data: $bel(x_{t-1}), u_t, z_t$
Result: $bel(x_t)$
foreach $x_t$ do
    $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$;
    $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$;
end
Return $bel(x_t)$
```

Derivation: measurement update

$$bel(x_t) = p(x_t \mid z_{1:t},\, u_{1:t})$$

$$= \frac{p(z_t \mid x_t, z_{1:t-1},\, u_{1:t})\, p(x_t \mid z_{1:t-1},\, u_{1:t})}{\underbrace{p(z_t \mid z_{1:t-1},\, u_{1:t})}_{:=\eta^{-1}}}$$

Bayes rule

$$= \eta\, p(z_t \mid x_t)\ \underbrace{p(x_t \mid z_{1:t-1},\, u_{1:t})}_{=\overline{bel(x_t)}}$$

Markov property

Derivation: correction update

$$\overline{bel}(x_t) = p(x_t \,|\, z_{1:t-1},\, u_{1:t})$$

$$= \int p(x_t \,|\, x_{t-1},\, z_{1:t-1},\, u_{1:t})\, p(x_{t-1} \,|\, z_{1:t-1},\, u_{1:t})\, dx_{t-1}$$

Total probability

$$= \int p(x_t \,|\, x_{t-1},\, u_t)\, p(x_{t-1} \,|\, z_{1:t-1},\, u_{1:t})\, dx_{t-1}$$

Markov

$$= \int p(x_t \,|\, x_{t-1},\, u_t)\, p(x_{t-1} \,|\, z_{1:t-1},\, u_{1:t-1})\, dx_{t-1}$$

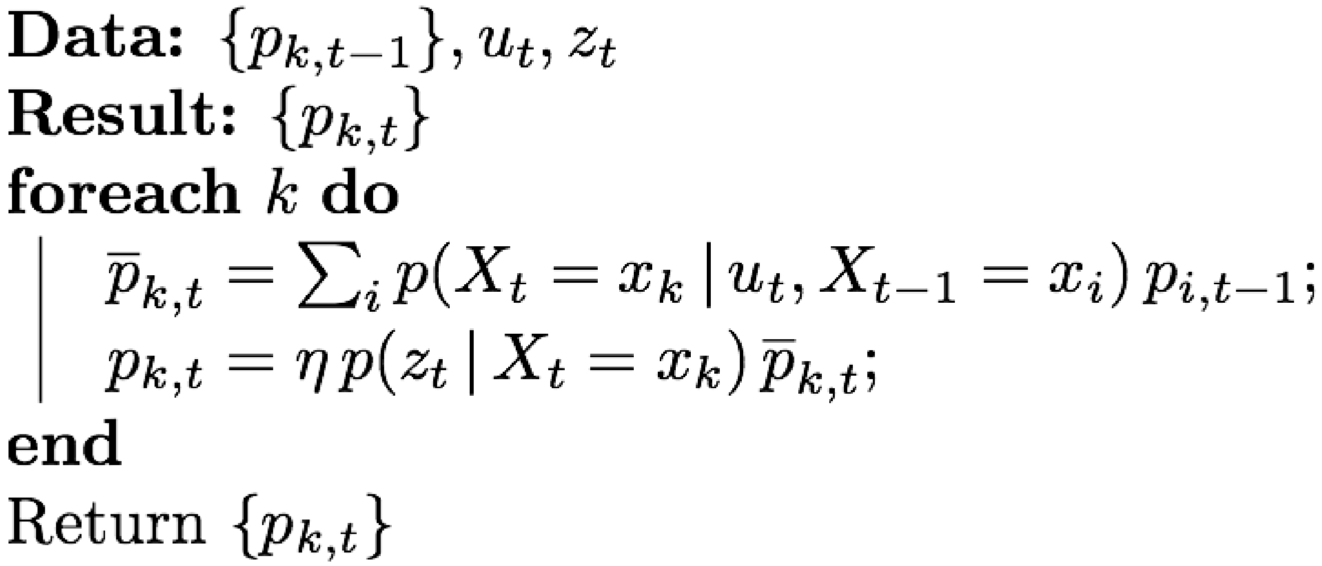
For general output feedback policies, u_t does not provide additional information on x_{t-1}

$$= \int p(x_t \,|\, x_{t-1},\, u_t)\, bel(x_{t-1})\, dx_{t-1}$$

Discrete Bayes' filter

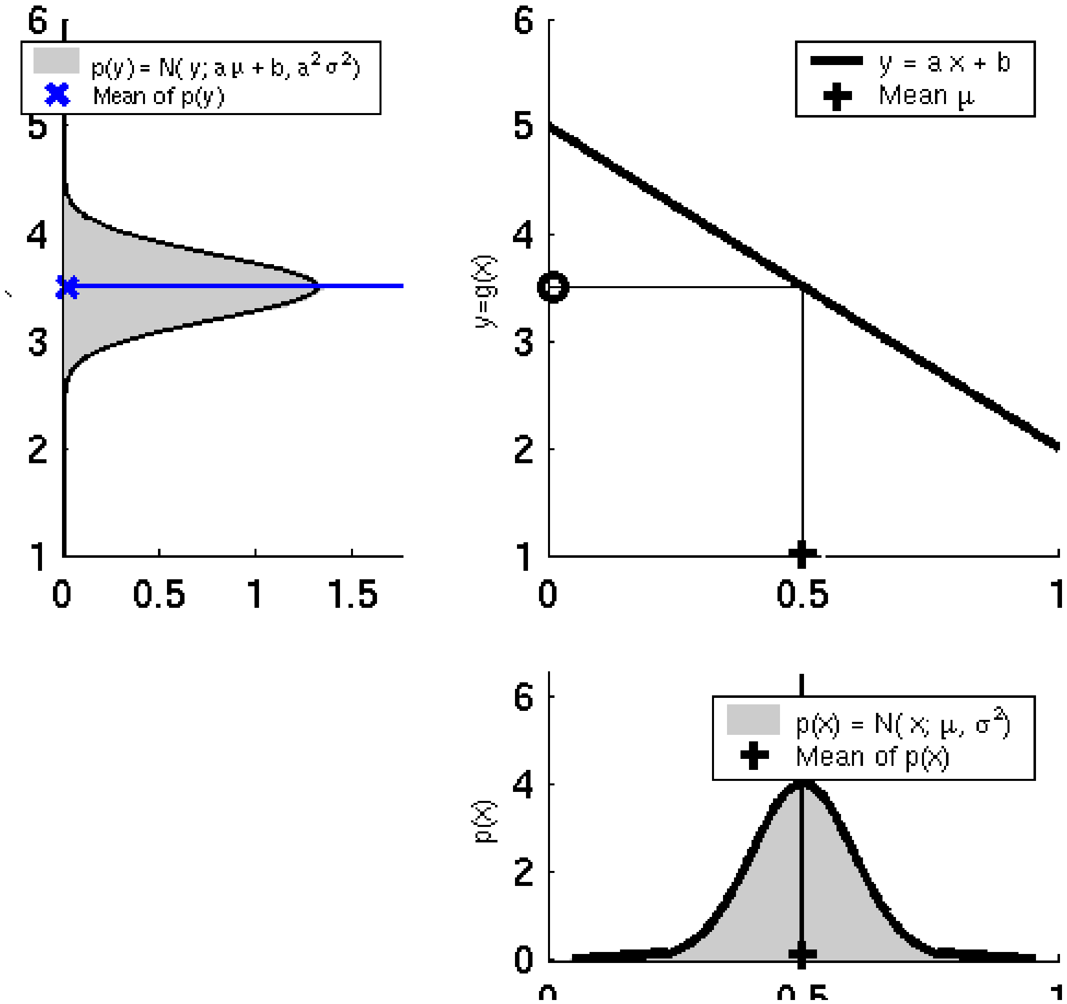
- Discrete Bayes' filter algorithm: applies to problems with *finite* state spaces

- Belief $bel(x_t)$ represented as pmf $\{p_{k,t}\}$



```
**Data:** $\{p_{k,t-1}\}, u_t, z_t$
**Result:** $\{p_{k,t}\}$
**foreach** $k$ **do**
    $\overline{p}_{k,t} = \sum_i p(X_t = x_k \,|\, u_t, X_{t-1} = x_i)\, p_{i,t-1}$;
    $p_{k,t} = \eta\, p(z_t \,|\, X_t = x_k)\, \overline{p}_{k,t}$;
**end**
Return $\{p_{k,t}\}$
```

Next time


p(y) = N(y; a μ + b, a² σ²)
Mean of p(y)
6
5
4
3
2
1
0
0.5
1
1.5
y = a x + b
Mean μ
y=g(x)
6
5
4
3
2
1
0
0.5
1
p(x) = N(x; μ, σ²)
Mean of p(x)
p(x)
6
4
2
0